# LOGARITHMS

A logarithm of a number to base *x* is the power to which *x* must be raised to give a number e.g.



The following 3 statements are equivalent

1. and
2. and log327 = 3
3. and

Logarithms appear in all sorts of calculations in engineering and science, business and economics. Before days of calculators, they were used to assist in the process of multiplication by replacing the operation of multiplication by addition similarly they enabled the operation of division to be replaced by subtraction.

**Laws of Logarithms**

1. 
2. (Change of base law)

**Example I**

Prove the following laws of logarithms

**Solution:**

1. Suppose and then equivalent logarithmic forms are

and

Using the first rule of indices,

Now the logarithmic form of the statement

is ,

but and

Putting this result together we have:

|  |
| --- |
| Therefore, (as required.) |

1. **For **

Suppose and with equivalent respective logarithmic forms

and .

|  |
| --- |
| as required |

1. **For**

Suppose or.

Suppose we raise both sides of to the power m

Using the rules of indices this can be written as;

Thinking of the quantity as a single term, the logarithmic form is









1. **For**

Let

…………………… (1)

Introducing on both sides of equation (1)







1. For

Let

**Example II**

Solve 

***Solution***



But

Let

Since,



**Example III**

Solve

**Solution:**

Let *m* =

log2*x* = -1

**Example IV**

Solve for *x*: 3log2 *x* – log*x*2 = 2

***Solution***

3log2 *x* – log*x* 2 = 2

But

Let

Since

**Example IV**

Solve

***Solution***

**Example V**

Prove that

***Solution***

**Example VI (UNEB Question)**

Given that . Prove that

***Solution***

……………………. (1)

……………………… (2)

Equating equation (1) and (2)

**Example VII**

If . Prove that

***Solution***

…………. (1)

But

Substituting for *m* = 4*a* in Eqn (1);

……………….. (2)

Since

…………………….. (3)

Substituting eqn. (3) in eqn. (2)

**Example VIII (UNEB Question)**

Prove that . Hence find log86 if log43 = 0.7925.

***Solution***

………………… (1)

But 





 ……………………….. (2)

substituting Eqn (2) in Eqn (1);

But,

**Example (UNEB Question)**

Solve log*x* 5 + 4log5 *x* = 4

***Solution***

log*x* 5 + 4log5 *x* = 4



Let *m* = log*x* 5



*m*2 + 4 = 4m

*m*2 – 4*m* + 4 = 0

(*m* – 2)2 = 0

*m* = 2

log*x*5 = 2

*x*2 = 5

**, 

**Example IX**

Solve 6log3 *x* + 6log27 *y* = 7

4log9 *x* + 4log3 *y* = 9

***Solution***

……………… (1)

……………….. (2)

From equation (1)

……………….. (3)

From equation (2)

………………. (4)

Let *A =* log3 *x*,

…………………… (5)

2A+ 4B = 9 …………………….. (6)

Solving equation 5 and 6 simultaneously

But,

*B* = log3 *y*

2 = log3 *y*

*y =* 32

*y*  = 9

, *y* = 9

**Example X**

Given that , show that *xy* = 16. Hence solve for x and y in the equations.

***Solution***

………………….. (1)

From eqn. (1)

But 

= 2







24 = *xy*

…………………………. (2)

From

*y*  = 10 – *x* ………………………… (3)

Substituting Eqn (3) in Eqn (2);

Substituting Eqn (3) in Eqn (1)

16 = *x*(10 – *x*)

16 = 10*x* – *x*2

*x*2 – 10*x* + 16 = 0

(*x* – 2)(*x* – 8) = 0

*x* = 2, *x* = 8

10 = *x* + *y*

If *x* = 2, 10 = 2 + *y*

*y* = 8

If *x* = 8, *y* = 2

**Example XI**

Solve for *x* and *y*;



***Solution***

1 = *x* – 3*y* + 2

*x* – 3*y* = -1 …………….……… (1)

…………………………. (2)

From eqn. (1)

……………… (3)

Substituting Eqn (3) in Eqn (2)

*x =* 17

*x* = -1, *x* = 17

**Example**

Solve the equation 

***Solution***



But 





2*y*2 – 6 – *y* = 0

(2*y* + 3)(*y* – 2) = 0



If *y* = 2 

42 = *x*

*x* = 16

If 

****

**Application of Indices**

**Example XIII**

Solve the equation 

***Solution***



Let

2*x* = 2-4

**Example IV**

Solve the following equations

***Solutions***

For



= 0.86135

For 5*x* = 50

*x* = 0

[Since ]

Let

But

1. 4*x* + 2 = 3 × 2*x*

Let



*x* = 1.465

**Example (UNEB Question)**

Solve the equations:

9*x* – 3*x* + 1 = 10

***Solution***

9*x* – 3*x* + 1 = 10

(32)*x* − 3*x*× 3 = 10

(3*x*)2 – 33*x* − 10 = 0

Let 3*x* = *y*

*y*2 – 3*y* – 10 = 0

(*y* – 5)(*y* + 2) = 0

*y* = 5, *y* = -2

3*x* = 5





*x* = 1.465

# ROOTS OF QUADRATIC EQUATIONS

Any equation of the form *ax*2 + *bx* + *c* = 0 where *a* ≠ 0 is called a quadratic equation.

**** is the formula used to solve quadratic equations.

*b*2 – 4*ac* is a discriminant and determines the nature of the roots.

**Nature of roots of quadratic Equations**

**** is the formula used to solve the equation *ax*2 + *bx* + *c* = 0, we see that

**Either**

**Or **

Therefore in general, a quadratic equation has two solutions (called roots).

**1. If *b*2 – 4*ac* is positive**

 can be evaluated and the equation has two real and distinct (different) roots.

***Illustration***

*x*2

*x*1

**2. If *b*2 – 4*ac* is zero**

 is zero; the equation is satisfied by only one value of *x*;  and we say that it has repeated roots or equal roots.

*y* = *ax*2 + *bx* + *c*

**3. If**  ***b*2 – 4*ac* is negative**

 has no real values; so the equation has no real roots.

To summarise the equation *ax*2 + *bx* + *c* = 0;

|  |
| --- |
| **Has two distinct roots if *b*2 – 4*ac* > 0**  **Has equal roots if *b*2 – 4*ac* = 0**  **Has no real roots if *b*2 – 4*ac* < 0; and *b*2 – 4*ac* is called a discriminant.** |

**Example I**

Determine the nature of roots of the following equations:

1. 4*x*2 – 7*x* + 3 = 0
2. *x*2 + *ax* + *a*2 = 0
3. *x*2 – *px* – *q* = 0

***Solutions***

**(a)** 4*x*2 – 7*x* + 3 = 0

Compare 4*x*2 – 7*x* + 3 = 0 with *ax*2 + *bx* + *c* = 0, it follows that *a* = 4, *b* = -7 and *c* = 3

The discriminant = *b*2 – 4*ac*

*b*2 – 4*ac*

(-7)2 – 4 × 4 × 3

= 49 – 48

= 1 > 0

Since *b*2 – 4*ac* >0,

The equation 4*x*2 – 7*x* + 3 = 0 has two real distinct roots

**(b) *x*2 + *ax* + *a*2 = 0.**

The discriminant is *b*2 – 4*ac*

*b*2 – 4*ac* = (*a*)2 – 4 × 1(*a*2)

= *a*2 – 4*a*2

= -3*a*2

Since *a*2 is positive irrespective of the value of *a*,

*b*2 – 4*ac* < 0. So the equation *x*2 + *ax* + *a*2 = 0 has no real roots.

**(c) *x*2 – *px* – *q*2 = 0**

Comparing *x*2 – *px* – *q*2 = 0 with *ax*2 + *bx* + *c* = 0, it follows that *a* = 1, *b* = *-p*, and *c* = -*q*2.

The discriminant is *b*2 – 4*ac*

*b*2 – 4*ac* = (-*p*)2 – 4(1)(-*q*2)

= *p*2 + 4*q*2

*p*2 + 4*q*2 > 0 irrespective of the values of *p* and *q.*

Since *b*2 – 4*ac* > 0,

*x*2 – *px* – *q*2 = 0 has two real distinct roots.

**Example II**

Determine the nature of the roots of the following equations:

1. *x*2 – 6*x* + 9 = 0
2. *x*2 – 2*x* + 1 = 0
3. *x*2 – 6*x* + 10 = 0
4. 4*x*2 – 12*x* – 9 = 0

***Solution***

**(a) *x*2 – 6*x* + 9 = 0**

Comparing *x*2 – 6*x* + 9 = 0 with *ax*2 + *bx* + *c* = 0 gives *a* = 1, *b* = -6, *c* = 9

The discriminant is b2 – 4*ac*

= (-6)2 – 4 × 1 × 9

= 36 – 36

= 0

Since *b*2 – 4*ac* = 0,

The equation *x*2 – 6*x* + 9 = 0 has repeated roots.

**(b) *x*2 – 2*x* + 1 = 0**

Comparing *x*2 – 2*x* + 1 = 0 with *ax*2 + *bx* + *c* = 0, it follows that *a* = 1, *b* = -2, *c* = 1

The discriminant is *b*2 – 4*ac*

= (-2)2 – 4 × 1 × 1

= 4 – 4

= 0

Since *b*2 – 4*ac* = 0,

** The equation *x*2 – 2*x* + 1 = 0 has repeated roots.

**(c) *x*2 – 6*x* + 10 = 0**

Comparing *x*2 – 6*x* + 10 = 0 with *ax*2 + *bx* + *c* = 0 gives *a* = 1, *b* = -6, *c* = 10

The discriminant is *b*2 – 4*ac* = 0

= (-6)2 – 4 × 1 × 10

= 36 – 40

= -4

Since *b*2 – 4*ac* < 0,

 The equation *x*2 – 6*x* + 10 = 0 has no real roots.

**(d) 4*x*2 – 12*x* – 9 = 0**

Comparing 4*x*2 – 12*x* – 9 = 0 with *ax*2 + *bx* + *c* = 0 gives a = 4, *b* = -12, *c* = -9.

The discriminant is b2 – 4*ac*

= (4)2 – 4 × 4 × (-9)

= 16 – 16 × -9

= 16 + 144

= 160

*b*2 – 4*ac* > 0

The equation 4*x*2 – 12*x* – 9 = 0 has two real distinct roots.

**Example III**

Find the values of *k* for which the following equations have equal roots.

1. 3*x*2 + *kx* + 12 = 0
2. *x*2 – 5*x* + *k* = 0

***Solution***

**(i) 3*x*2 + *kx* + 12 = 0**

For a quadratic equation to have equal roots,

*b*2 = 4*ac*

Comparing 3*x*2 + *kx* + 12 = 0 with *ax*2 + *bx* + *c* = 0 gives *a* = 3, *b* = *k* and *c* = 12

*b*2 = 4*ac*

 (*k*)2 = 4 × 3 × 12

*k*2 = 144

*k* = ±12

*k* = 12, *k* = -12

**(ii) *x*2 – 5*x* + *k* = 0**

For the equation *x*2 – 5*x* + *k* = 0 to have real roots,

*b*2 = 4*ac*

 (-5)2 = 4 × 1 × *k*

25 = 4*k*

*k* = 

**Example IV**

Prove that *kx*2 + 2*x* – (*k* – 2) = 0 has real roots for any values of *k.*

***Solution***

*k*2 + 2*x* – (*k* – 2) = 0

Comparing *kx*2 + 2*x* – (*k* – 2) = 0 with *ax*2 + *bx* + *c* = 0 gives *a* = *k*, *b* = 2, *c* = -(*k* – 2)

The discriminant is *b*2 – 4*ac*

= (2)2 – 4 × *k*[-(*k* – 2)]

= 4 + 4*k*(*k* – 2)

= 4 + 4*k*2 – 8k

= 4*k*2 – 8*k* + 4

= 4(*k*2 – 2*k* + 1)

= 4(*k* – 1)2

Since 4(*k* – 1)2 > 0, *kx*2 + 2*x* – (*k* – 2) has two real distinct roots for any values of *k*.

**Example V**

Find the range of values *k* can take for 9*x*2 + *kx* + 4 = 0 to have two real distinct roots.

***Solution***

Comparing 9*x*2 + *kx* + 4 = 0 with *ax*2 + *bx + c* = 0 gives *a* = 9, *b* = *k*, *c* = 4.

The discriminant is *b*2 – 4*ac*

= (*k*)2 – 4 × 9 × 4

= *k*2 – 144

For two distinct real roots, *b*2 – 4*ac* > 0

*k*2 – 144 > 0

(*k* + 12)(*k* – 12) > 0

For the boundary conditions, *k* = -12, *k* = 12

|  |  |  |  |
| --- | --- | --- | --- |
|  | *k* < -12 | -12 < *k* < 12 | *k* > 12 |
| *k* + 12 | –ve | +ve | +ve |
| *k* – 12 | –ve | –ve | +ve |
| (*k* + 12)(*k* – 12) | +ve | –ve | +ve |

For 9*x*2 + *kx* + 4 = 0 to have real roots, the product

(*k* + 12)(*k* – 12) must be positive.

*k* < -12 and *k* > 12 are ranges of values for which

9*x*2 + *kx* + 4 = 0 has real distinct roots.

**Example VI (UNEB Question)**

Find the value of *k* for which the equation  = *k* has repeated roots. What are the repeated roots?

***Solution***

 = *k*

*x*2 – *x* + 1 = *kx* – *k*

*x*2 – *x* – *kx* + 1 + *k* = 0

*x*2 – (*k* + 1)*x* + *k* + 1 = 0

For repeated roots, *b*2 = 4*ac*

Comparing *x*2 – (*k* + 1)*x* + *k* + 1 = 0 with

a*x*2 + *bx* + c = 0 gives a = 1, *b* = -(*k* + 1), *c* = *k* + 1)

*b*2 = 4*ac*

[-(*k* + 1)]2 = 4 × 1(*k* + 1)

(*k* + 1)2 = 4*k* + 4

*k*2 + 2*k* + 1 = 4*k* + 4

*k*2 – 2*k* – 3 = 0

(*k* – 3)(*k* + 1) = 0

*k* = 3 **OR** *k* = -1

If *k* = 3,

*x*2 – (*k* + 1)*x* + *k* + 1 = 0

*x*2 – 4*x* + 4 = 0

(*x* – 2)2 = 0

*x* = 2, *x* = 2

When *x* = 3, the repeated roots are *x*=2 and *x* = 2

If *x* = -1;

*x*2 – (*k* + 1)*x* + *k* + 1 = 0

*x*2 = 0

*x* = 0, *x* = 0

When *k*= -1, the repeated roots are *x*=0 and *x* =0.

**Maximum and Minimum values of a quadratic function**

Consider *y* = *ax*2 + *bx* + *c*

Using the method of completing squares, the quadratic equation can be reduced to:

1. *a*(*x* – *p*)2 + *q*
2. *q* – *a*(*x* – *p*)2

**(i)** Let *y* = *a*(*x* – *p*)2 + q

Since (*x* – *p*)2 is never negative, the least value of *y* occurs when (*x* – *p*)2 = 0

**(ii)** For *y* = *q* – *a*(*x* – *p*)2;

Since (*x* – *p*)2 is never negative

 The maximum value of *y* is *q.*

**Examples**

Find the greatest or least values of the following functions:

1. *x*2 – 2*x* + 5
2. 5 – 4*x* – *x*2
3. *x*2 – 3*x* + 5
4. 2*x*2 – 4*x* + 5
5. 7 + *x* – *x*2
6. *x*2 – 2
7. 2*x* – *x*2

***Solution***

**(a) *x*2 – 2*x* + 5**

By completing squares,

*x*2 – 2*x* + 5 = *x*2 – 2*x* + + 5

= *x*2 – 2*x* + 1 – 1 + 5

= (*x* – 1)2 + 4

*y* = *x*2 – 2*x* + 5

*y* = 4 + (*x* – 1)2

The least value of *y* is 4 and it occurs when (*x*–1)2 =0

**(b) 5 – 4*x* – *x*2**

By completing squares;

5 – 5*x* – *x*2 = 5 – (*x*2 + 4*x*)

= 5 – (*x*2 + 4*x* + 4) – −4

= 5 – (*x* + 2)2 + 4

= 9 – (*x* + 2)2

*y* = 9 – (*x* + 2)2

The greatest value is *y* = 9 and it occurs when

(*x* + 2)2 = 0

**(c) *x*2 – 3*x* + 5**

By completing squares;

*x*2 – 3*x* + 5 = *x*2 – 3*x* +  + 5

= 

*y* =

The least value of *y* is  and it occurs when 

**(d) 2*x*2 – 4*x* + 5**

2(*x*2 – 2*x*) + 5

2(*x*2 – 2*x* + 1) – 2 + 5

3 + 2(*x* – 1)2

*y* = 3 + 2(*x* – 1)2

The least value of *y* is 3 and it occurs when

2(*x* – 1)2 = 0

**(e) 7 + *x* – *x*2**

7 – (*x*2 – *x*)

7 – (*x*2 – *x* + ) – 

7 – 



The greatest value of *y* is  and it occurs when  = 0

**(f) *x*2 − 2**

The least value of *y* is -2 and it occurs when *x*2 = 0

**(g) 2*x* – *x*2**

*y* = 2*x* – *x*2

*y* = -(*x*2 – 2*x*)

By completing squares;

*y* = -(*x*2 – 2*x* + 1) – −1

*y* = -(*x* – 1)2 + 1

*y* = 1 – (*x* – 1)2

The greatest value of *y* = 1 and it occurs when *x* = 1

## Sum & Product of the roots of Quadratic Equations

Consider the equation *ax*2 + *bx* + *c* = 0

 *x*2 +  = 0 ……………………. (i)

Suppose *α* and *β* are the roots of the equation

*x*2 +  = 0

We can use *α* and *β* to form an algebraic equation in which the unknown quantity *x* is satisfied by putting *x* = *α* or *x* = *β*.

*x* = α or *x* = β

*x* – α = 0 or *x* – β = 0

(*x* – α)(*x* – *β*) = 0

*x*2 – *βx* – *αx* + *αβ* = 0

*x*2 – (α + *β*)*x* + *αβ* = 0 ……………………… (ii)

Eqn (i) and Eqn (ii) have the same roots, must be precisely the same equation written in two different ways.

Equating coefficients of the same monomials in Eqn (i) and Eqn (ii);

 -(α + *β*) = **

(*α* + *β*) = 

Similarly, *αβ* = 

For a quadratic function with roots *α* and *β*,

Sum of roots = *α + β* = **

Product of roots *αβ* = 

*α*2 + *β*2 = (*α +β*)2 – 2*αβ*

*α*3 *+ β*3 = (*α* + *β)*3 – 3*αβ* (*α* + *β*)

α3 + *β*3 = (*αβ*)3

(*α* – *β*) = 

*α* – *β*  = 

*α* – *β* = 

*α* – *β* = 

(*α* – *β*) = .





|  |
| --- |
| **The following are important formulae used under roots of quadratic equations.**  **(*α*2 +*β*2) = (*α* + *β*)2 – 2*αβ***  ***α*3 + *β*3 = (*α* + *β*)3 – 3*αβ*(*α* + *β*)**  ***α* – *β* =** |

**Example I**

If *α* and *β* are roots of the equation *x*2 + 8*x* + 1 = 0, find the values of

1. *αβ*
2. *α* + *β*
3. *α*2*β* + *αβ*2
4. *α*2 + *β*2

***Solution***

**(a) *x*2 + 8*x* + 1 = 0**

Comparing *x*2 + 8*x* + 1 = 0 with *ax*2 + *bx* + *c* = 0 gives

*a* = 1, *b* = 8, *c* = 1

*α β* = 

*αβ*  =  = 1

**(b) *α* + *β* =** .

*α* + *β* 

*α* + *β = -*8

**(c) *α*2*β* + *α β*2**

= *α β*(*α + β*)

= 1(-8)

= -8

∴ *α*2*β*+ *α β*2 = -8

**(d) *α*2 + *β*2 = (*α + β*)**2 – 2 *α β*

*α* + *β* = -8

*α β* = 1

*α*2 + *β*2 = (-8)2 – 2 × 1

= 64 – 2

= 62

**Example II**

If *α* and *β* are roots of the equation *x*2 – *x* – 3 = 0, state the values of *α* + *β* and *αβ* and find the values of:

1. *α*2 + *β*2
2. (*α* – *β*)2
3. *α*3 + *β*3

***Solution***

Comparing *x*2 – *x* – 3 = 0 with *ax*2 + *bx* + *c* = 0 gives *a* = 1, *b* = -1, *c* = -3

*α* + *β* = 

 *α* + *β* =  = 1

*α β* = 

*α β* =  = -3

**(a)** *α*2+ *β*2

(*α + β*)2 – 2*αβ*

= (1)2 – 2(-3)

= 1 – –6

= 7

**(b)** (*α* – *β*)2 = *α*2 – 2*αβ* + *β*2

= *α*2 + *β*2 – 2*αβ*

But *α*2 + *β*2 = (*α+β*)2– 2*αβ*

= (*α* + *β*)2 – 4*αβ*

(*α* - *β*)2 = (*α* + *β*)2 – 4*αβ*

= 12 – 4 × (-3)

= 13

**(c**) *α*3 + *β*3 = (*α + β*)3 – 3 *αβ*(*α + β*)

*α*3 + *β*3= (1)3 – 3(-3)(1)

= 1 + 9

= 10

**Example III**

If *α* and *β* are roots of the equation 2*x*2 – 5*x* + 1 = 0, find the values of:

1. ***α + β***
2. *α β*
3. *α*2 + 3*αβ* + *β*2
4. *α*2 – 3*αβ + β*2.
5. *α*3*β* + *αβ*3
6. 

***Solution***

Comparing 2*x*2 – 5*x* + 1 = 0 with *ax*2 + *bx* + *c* = 0 gives

*a* = 2, *b* = -5 and *c* = 1

*α + β* = 

*α + β* = 

 *α + β* = 

*α β* = 

 *α β* = 

**(c) *α*2 + 3*αβ* + *β*2**

= (*α*2 + *β*2) + 3*αβ*

= (*α* + *β*)2 – 2*αβ* + 3*αβ*

= (*α* + *β*)2+ *αβ*

= 

= 

= 

**(d) *α*2** – 3*αβ* + *β*2

*α*2 + *β*2– 3*αβ*

(*α* + *β*)2 – 2 *αβ* – 3*αβ*

(*α* + *β*)2 – 5*αβ*

= 

= 

= 

**(d) *α*3** *β* + *αβ*3

= *α β* (*α*2 + *β*2)

= *α β*[(*α* + *β*)2 – 2 *αβ*]

= 



**(f)**  

= 

= 

= 5

**Example IV**

If *α* and *β* are roots of the equation 6*x*2 + 2*x* – 3 = 0, find the values of:

1.  (b) 

(c)  (d) 

(e) *α*3**+** *β*3 (f) 

***Solution***

Comparing 6*x*2 + 2*x* – 3 = 0 with *ax*2 + *bx* + *c* = 0,

gives *a* = 6, *b* = 2, *c* = -3





*α* + *β* = 

*α + β* = 

*αβ* = 

*αβ* = 





**(b)**  



= 





**(c)** 





**(c)** *α*3 + *β*3 = (*α + β*)3 – 3 *α β*(*α* + *β*)



**(f)** 





**Example V**

If *α*2 and *β*2 are roots of the equation *x*2 – 21*x* + 4 = 0, and *α* and *β* are both positive, find *α β* and *α* + *β.*

***Solution***

Comparing *x*2 – 21*x* + 4 = 0 with *ax*2 + *bx* + *c* = 0 gives

*a* = 1, *b* = -21, *c* = 4

*α*2 + *β*2 = 

= 

= 21

*α*2 *β*2 = 4

*α*2 + *β*2 = (*α* + *β*)2 – 2 *α β*

(α *+ β*)2 – 2*αβ* = 21

(α + β)2 - 2 × 2 = 21

(α + β)2 = 25

(α + β) = 5

**Example VI**

Write down the equation whose roots are:

(a) 3, 4 (b) -2, ½

(c) ,  (d) , 0

(e) *a*2, *a*2 (f) -(*k* + 1), *k*2 – 3

(g) , 

***Solution***

|  |
| --- |
| **Any quadratic equation is given by**  ***x*2 – (sum of roots)*x* + product of roots = 0** |

**(a)**  *x* = 3, 4

Sum of roots = 3 + 4

= 7

Product of roots = 3 × 4

= 12

*x*2 – (sum of roots)*x* + product = 0

*x*2 – (7)*x* + 12 = 0

*x*2 – 7*x* + 12 = 0

**(b)** *x* = -2, *x* = 

Sum of roots = -2 + **

Sum of roots = 

Product of roots = -1

*x*2 – (sum of roots)*x* + product = 0

*x*2 – *x* + -1 = 0

*x*2 + *x* – 1 = 0

2*x*2 + 3*x* – 2 = 0

**(c)** *x* = , *x* = 

Sum of the roots = 

= 

Product of the roots = 

= 

*x*2 – (sum of roots)*x* + product of roots = 0

*x*2 – *x* +  = 0

15*x* + *x* – 2 = 0

**(d)** *x* = , *x* = 0

Sum of roots =  + 0 = 

Product of roots = 0

*x*2 – (sum of roots)*x* + product of roots = 0

*x*2 – *x* = 0

4*x*2 + *x* = 0

**(e)** *x* = *a*2, *x* = *a*2

Sum of the roots = *a*2 + *a*2

= 2*a*2

Product of the roots = *a*2 × *a*2

= *a*4

*x*2– (sum of roots)*x* + product of roots = 0

*x*2 **–** (2*a*2)*x* + *a*4 = 0

*x*2 – 2*a*2*x* + *a*4 = 0

**(f)** -(*k* + 1), *k*2 − 3

Sum of roots = -*k* + 1 + *k*2 – 3

= *k*2 – *k* – 2

Product of roots = -(*k* + 1)(*k*2 – 3)

= -(*k*3 – 3*k* + *k*2 – 3)

Product of roots = -*k*3 + 3*k* – *k*2 + 3

*x*2 – (sum of roots)*x*  + product of roots = 0

*x*2 – (*k*2 – *k* – 2)*x* + 3*k* + 3 – *k*2 – *k*3 = 0

**(g)** *x* = , *x* =

Sum of roots = 

= 

Product of the roots = 

 = 0

*a*2*bx*2 – *a*(*b*2 + *a*2*c*)*x* + *c*2 = 0

**Example VII**

The roots of the equation *x*2 – 2*x* + 3 = 0 are *α* and *β*. Find the equation whose roots are:

1. *α* + 2, *β* + 2
2. , 
3. *α*2 *, β*2
4. , 

***Solution***

*x*2 – 2*x* + 3 = 0

Comparing *x*2 – 2*x* + 3 = 0 with *ax*2 + *bx* + *c* = 0 gives *a* = 1, *b* = -2, and *c* = 3.

*α + β* = = 2

*αβ* =  = 3

New sum of roots = *α* + 2 + *β* + 2

= *α* + *β* + 4

= 2 + 4

= 6

New product of the roots = (*α +* 2)(*β* + 2)

= *αβ* + 2*α* + 2*β* + 4

= *αβ* + 2(*α + β*) + 4

= 3 + 2(2) + 4

= 3 + 4 + 4

= 11

Any quadratic equation is given by:

*x*2 – (sum of roots)*x* + product of roots = 0

*x*2 – (2)*x* + 11 = 0

*x*2 − 2*x* + 11 = 0

**(b)** , 

*α + β* = 2

*α β* = 3

New sum of roots = 

= **

New product of roots = 



*x*2 – (sum of the roots)*x* + product of roots = 0.

*x*2 – *x* + ** = 0

3*x*2 – 2x + 1 = 0

**(c)** *α*2, *β*2

*α* + *β* = 2

*α β* = 3

New sum of the roots = *α*2 *+ β*2

= (*α + β*)2 *–* 2 *α β*

= 22 – 2(3)

= 4 – 6

= -2

New product of roots = *α*2 *β*2

= (*αβ*)2

= 32

= 9

*x*2 – (sum of the roots)*x* +product of the roots = 0

*x*2 – (-2)*x* + 9 = 0

*x*2 + 2*x* + 9 = 0

**(d)** **, **

*α* + *β* = 2; *α β* = 3

New sum of roots = ******

******

******

****

New product of roots = 

= 1

*x*2 – (sum of roots)*x* + product of roots = 0

*x*2 – *x* + 1 = 0

3*x*2 + 2*x* + 3 = 0

**(e)** *α – β*, *β* – *α*

*α* + *β* = 2, *αβ* = 3

New sum of the roots = *α – β* + *β* – *α*

= *α* + *β* – (*α* + *β*)

= 0

New product of the roots = (*α – β*)(*β – α*)

= *αβ – α*2 – *β*2 + *αβ*

= 2*αβ* – [(*α*2+ *β*2)]

= 2*αβ* – [(*α + β*)2 – 2*αβ*]

= 4*αβ* – (*α* + *β*)2

= 4 × 3 – (22)

= 8

*x*2 – (sum of the roots)*x* + product of roots = 0

*x*2 – (0)*x* + 8 = 0

*x*2 + 8 = 0

**Example VIII**

The roots of the equation 2*x*2 + 7*x* – 3 = 0are *α* and *β*. Find the equation whose roots are  and**.**

***Solution***

Sum of the roots = *α* + *β* = **

= 

Product of the roots = *α β* = 

New sum of the roots = 

= *α + β* + 

= *α + β* + 







New product of roots = 

= *α β* + 5 + 5 + 

= *α β* +  + 10

=  + 10



*x*2 – (sum of roots)*x* + product of roots = 0

*x*2 – *x* +  = 0

6*x*2 + 91*x* – 49 = 0

**Example IX**

Given that *α* and *β* are roots of the equation

4*x*2 + 7*x* – 5 = 0. Find the equation whose roots are 2*α* – 1 and 2*β* – 1

***Solution***

Comparing 4*x*2 + 7*x* – 5 = 0 with *ax*2 + *bx* + *c* = 0 gives

*a* = 4, *b* = 7, and *c* = -5

Sum of the roots = *α + β* = 

*****α + β* = ****

Product of the roots = *α β* = ****



New sum of roots = 2 *α –* 1 + 2 *β* – 1

= 2(*α* +*β*) – 2

=



New product of roots = (2*α* – 1)(2 *β –* 1)

= 4*αβ* – 2*α* – 2*β* + 1

= 4*αβ* – 2(*α + β*) + 1



= -5 +  + 1



*x*2 – (sum of roots)*x* + product of roots = 0

*x*2  = 0

*x*2 + 11*x* – 1 = 0

**Example (UNEB Question)**

If the roots of the equation *x*2 + 2*x* + 3 = 0 are *α* and *β*, form an equation whose roots are *α*2– *β* and *β*2 – *α.*

***Solution***

Comparing *x*2 + 2*x* + 3 = 0 with *ax*2 + *bx* + *c* = 0 gives

*a* = 1, *b* = 2 and *c* = 3

Sum of roots = *α* + *β* = 

Product of the roots = 

 = -2

 = 3

New sum of the roots = *α*2 – *β* + *β*2 – *α*

= *α*2 + *β*2 – (*α + β*)

= (*α + β*)2 *–* 2*αβ* – (*α + β*)

= (-2)2 – 2(3) – (-2)

= 4 – 6 + 2

= 0

New product of roots = (*α*2– *β*)(*β*2– *α*)

= *α*2*β*2– *α*3 *– β*3+ *αβ*

= (*α β*)2 – (*α*3 *+ β*3) + *αβ*

= (*α β*)2 – [(*α* + *β*)3 **–** 3 *αβ* (*α* + *β*)] + *αβ*

= 32 – [(-2)3 – 3(3)(-2)] + 3

= 9 – [-8 + 18] + 3

= 9 – [10] + 3

= 2

*x*2 – (sum of the roots)*x* + product of roots = 0

*x*2 – (0)*x* + 2 = 0

*x*2+ 2 = 0

**Example (UNEB Question)**

If *α* and *β* are roots of the equation *x*2 – *px* + *q* = 0, find the equation whose roots are ** and ****

***Solution***

Comparing *x*2 – *px + q* = 0 with *ax*2 + *bx* + *c* = 0 gives

*a* = 1, *b* = *-p*, *c* = *q*

Sum of the roots = *α + β* = 

= 

Product of the roots = *α β* = 

New sum of the roots = 





New product of the roots = **

**





*x*2 – (sum of the roots)*x* + product of the roots = 0

 = 0

*qx*2 – (*p*2*q* – *p* – 2*q*2)*x* + *q*3 – *p*3 + 3*pq* + 1 = 0

**Example (UNEB Question)**

If *α* and *β* are roots of the equation *ax*2 + *bx* + *c* = 0, express (*α* – *β*) (*β –* 2*α*) in terms of *a*, *b* and *c*. Hence deduce the condition for the root to be twice the other.

***Solution***

*α* + *β* = 

*α β* = 

(*α* – 2*β*)(*β* – 2*α*)

*αβ –* 2*α*2 – 2*β*2 **+** 4*αβ*











****(*α –* 2 *β*)(*β* – 2 *α*)

For one root to be twice the other, (*α –* 2*β*)(*β* – 2*α*) = 0



9*ac* = 2*b*2

**Example (UNEB Question)**

Given that *α* and *β* are roots of the equation *x*2 + *px* + *q* = 0, express (*α – β*2)(*β* – *α*2) in terms of *p* and *q.* Deduce that for one root to be a square of another root, *p*3 – 3*pq* + *q*2 + *q* = 0

***Solution***

Sum of the roots = *α* + *β* = 

*α* + *β* = *-p*

*α β* = *q*

(*α* + *β*2)(*β* – *α*2)

= *α β* – *α*3 – *β*3 + (*αβ*)2



= *q* – [(-*p*3 – 3*q*(-*p*)] + *q*2

= *q* + *p*3 – 3*pq* + *q*2

For one root to be a square of the other,

(*α – β*2)(*β – α*2) = 0

*α = β*2, *β* = *α*2

(*α – β*2)(*β – α*2) = *q* + *p*3 – 3*pq* + *q*2

*******p*3 *–* 3*pq* + *q* + *q*2 = 0

**Example (UNEB Question)**

Given that *α* and *β* are roots of the equation *ax*2 +*bx* +*c*= 0, determine the equation whose roots are *α + β* and *α*3 + *β*3.

***Solution***

Sum of roots = *α + β* = **

Product of roots = *αβ* = 

New sum of roots = (*α* + *β*) + *α*3 + *β*3





New product of roots = (*α + β*)(*α*3 + *β*3

= (*α* + *β*)[(*α* + *β*)3 – 3*αβ* (*α* + *β*)]

= 

= 

= 

= 

= 

*x*2− (sum of roots)*x* + product of roots = 0



*a*4*x*2 + (*a*2*b* + *b*3 – 3*abc*)*x* + *b*4 + 3*ab*2*c* = 0

**Example (UNEB Question)**

Given that equations *y*2 + *py* + *q* = 0 and *y*2 + *my* + *k* = 0 have a common root, show that (*q* – *k*)2 = (*m* – *p*)(*pk* – *mq*)

***Solution***

Let the common root be *α*.

 *α*2 *+ pα* + *q* = 0 ……………………. (i)

*α*2 + *mα* + *k* = 0 …………………… (ii)

Eqn (i) – Eqn (ii);

**(*p – m*)*α* + *q* – *k* = 0

-(*m – p*)*α* + *q* – *k* = 0

 ……………………….. (iii)

Substituting Eqn (iii) in Eqn (i);

* + q =* 0

(*q* – *k*)2 + *p*(*m* – *p*)(*q* – *k*) + *q*(*m* – *p*)2 = 0

(*q* – *k*)2 + (*m* – *p*)(*pq* – *pk* + *qm* – *pq*) = 0

(*q* – *k*)2 + (*m* – *p*)(*qm* – *pk*) = 0

(*q* – *k*)2 – (*m* – *p*)(*pk* – *qm*) = 0

(*q* – *k*)2 = (*m* – *p*)(*pk* – *qm*)

**Example**

If *α* and *β* are roots of *px*2 + *qx* + *r* = 0, form an equation with algebraic integral coefficients whose roots are **** and ****

***Solution***

*α + β* = 

* α + β* = 

*αβ* = **

 *αβ* = ****

New sum of the roots = **

= 

= 

= 

= 

= ****

**=** **

New product of the roots = 

= 

= 

= 

=

*x*2 – (sum of the roots)*x* + product of the roots = 0

 = 0

(*p* – *q* + *r*)*x*2 – 2(*p* – *r*)*x* + *p* + *q* + *r* = 0

## Revision Exercise

1. The roots of the equation 4*x*2 + 4*x* – 1 = 0 are *α* and *β.* Find the values of: (a) 

(b) **

1. If *α* and *β* are roots of the equation 3*x*2 – 6*x* + 2 = 0. Find

(a) *α*2 – 3*αβ* + *β*2

(b) *α*3*β* + *αβ*3

(c) 

1. If *α* and *β* are roots of the quadratic equation *x*2 – 2*x* – 5 = 0. Find the quadratic equation whose roots are:

(a) *α –* 4, *β* – 4

(b) **,** 

(c) , **

1. The roots of the equation 3*x*2 – 8*x* + 2 = 0 are *α* and *β*. Find an equation whose roots are  and *.*
2. If the roots of the equation *ax*2 + *bx* + *c* = 0 differ by 4, show that  = 4*a* + *c*.
3. Prove that if the roots of the equation *ax*2 + *bx* + *c* = 0 is three times the other, then 3*b*2 = 16*ac*.
4. The roots of the equation *x*2 + 2*px* + *q* = 0 differ by 8. Show that *p*2 – 16 = *q*.
5. The roots of the equation *x*2 + 2*x* + *k* are *β* and *β* – 1. Find the value of *k*.
6. The roots of the equation *ax*2 + *bx* + *c* = 0 is a square of the other. Prove that *c*(*a* – *b*)3 = *a*(*c* – *b*)3.
7. If *α* and *β* are roots of the equation *px*2 + *qx* + *r* = 0, form an equation with integral coefficients whose roots are  and .
8. Given that *α* and *β* are roots of the equation 2*x*2 – 8*x* + 2 = 0, show that *α*3 + *β*3 = 52. Hence that *α*6 + *β*3 = 27.
9. Find the relationship between *p*, *q* and *r* if the roots of the equation *px*2 + *qx* + *r* = 0 double each other. Show that 
10. If the roots of the equation 2*x*2 – 3*x* – 1 = 0 are *α* and *β*, find the value of *α*2 + *β*2 and hence form the equation whose roots are *α*2 and *β*2.
11. Given that *α* is a common root of the equations

*x*2 – 2*x* – *k* = 0 and *x*2 – 5*x* + 2*k* = 0, where *k* ≠ 0. Find the numerical values of *k* and *α*.

1. In the equation *m*2*x*2 + 2*mnx* + *n*2 + 1 = 0, *m* and *n* are constants which are real numbers. Show that the equation has no real roots for any values of *m* and *n*.
2. The roots of the quadratic equation *ax*2 + *bx* + *c* = 0 are *α* and *β*. Write down the expression for (*α* + *β*) and *αβ*. Express in terms of *α* and *β*

(i)  (ii) 

(b) The roots of the equation 2*x*2 – 3*x* + 4 = 0 are *α* and *β*. Prove that  and  are also roots of the equation.

1. Solve the equation 22(*x* + 1) – 5 × 2*x* + 1 = 0
2. Solve the simultaneous equations

2*x* + *y* + 3 = *x* + *y* + 2 = 2*x*2 – 11*y*2 + 3

1. The roots of the equation *x*2 + *ax* + *b* = 0 is the square of the other. Find the roots in terms of *a* and *b*.
2. The roots of the equations 2*x*2 – 3*x* + 5 = 0 are *α* and *β.* And the roots of the equation *px*2 + *x* = *q* = 0 are

*α* – 1 and *β* – 1. Find the value of *p* and *q*.

1. If *α* and *β* are the roots of the equation 2*x*2 – *x* = 5, find the equation whose roots are *α* + 2*β* and *β* + 2*α*.
2. If *α* and *β* are roots of the equation 2*x*2 – 3*x* – 4 = 0, find the equation whose roots are  and .
3. If the roots of the equation *x*2 – 5*x* + 1 = 0 are *α* and *β*, form an equation with roots *α* + 3*β*, 3*α* + *β.*
4. If *α* and *β* are roots of the equation 3*x*2 – 3*x* – 1 = 0, form an equation whose roots are  and .
5. If *α* and *β* are roots of the equation 3*x*2 + *x* + 2 = 0,

(a) Evaluate 

(b) Find the equation whose roots are  and 

(c) Show that 27*α*4 = 11*α*  + 10

1. The roots of the equation *x*2 + 6*x* + *c* = 0 differ by 2*n*, where *n* is real and non-zero. Show that *n*2 = 9 – *c*. Given that the roots have opposite signs, find the set of all possible values of *n*.
2. Prove that the equation *x*(*x* – 2*p*) = *q*(*x* – *p*) has real roots for all values of *p* and *q*. If *p* = 3, find the non-zero value for *q.*
3. If the roots of the equation *x*2 + *bx* + *c* are *α* and *β* and the roots of the equation *x*2 + λ*bx* + λ2*c* = 0 are γ and *δ*. Show that the equation whose roots are *α*γ+ *βδ* and *αδ* + *β*γis *x*2 – λb2*x* + 2λ2*c*(*b*2 – 2*c*) = 0
4. The roots of the quadratic equation *x*2 – *px* + *q* = 0 are *α* and *β*. Determine the equation having the roots

*α*2 + *β*-2 and *β*2 + *α*-2.

1. Prove that the roots of the equation

(γ + 3)*x*2 + (6 – 2 γ)*x* + γ – 1 = 0 are real if and only if γ is not greater than . Find the values of γ if one root is six times the other.

1. Form the equation whose roots are the cubes of the roots of the equation *x*2 – 3*x* + 4 = 0.
2. Show that if the equations *x*2 + *bx* + c = 0 and

*x*2 + *px* + q = 0 have a common root, then

(*c* – *q*)2 = (*b* – *p*)(*cp* – *bq*)

1. (i) Write *x*2 + 6*x* + 16 in the form (*x* + *a*)2 + *b*, where *a* and *b* are integrals to be found.

(ii) Find the minimum values of *x*2 + 6*x* + 16 and state the value of *x* for which this minimum value occurs.

1. The roots of the equation 2*x*2 + 3*x* – 4 = 0 are *αβ*. Find the values of: (a) *α*2 + *β*2

(b) 

(c) (*α +* 1)(*β* + 1)

(d) 

1. If the roots of the equation 3*x*2 – 5*x* + 1 = 0 are *α* and *β*, find the values of:

(a) *αβ*2+ *α*2*β*  (b) *α*2 – *α β* + *β*2

(c) *α*3 + *β*3 (d)  **

1. The equation 4*x*2 + 8*x* – 1 = 0 has roots *α* and *β*. Find the values of:

(a)  (b) (*α* – *β*)2

(c) *α*3*β* + *αβ*3 (d) 

1. If the roots of the equation *x*2 – 5*x* – 7 = 0 are *α* and *β*, find the equations whose roots are:

(a) *α*2, *β*2 (b) *α* + 1, *β* + 1

(c) *α*2 *β*, *α β*2

1. The roots of the equation 2*x*2 – 4*x* + 1 are *α* and *β*. Find the equations with integral coefficients whose roots are:

(a) *α* – 2, *β* – 2 (b)  (c) 

1. Find the equation with integral coefficients whose roots are the squares of the roots of the equation

2*x*2 + 5*x* – 6 = 0

1. The roots of the equation *x*2 + 6*x* + *q* = 0 are *α* and

*α* – 1. Find the values of *q*.

1. The roots of the equation *x*2 – *px* + 8 = 0 are *α* and

*α* + 2. Find the two possible values of *p*.

1. The roots of the equation *x*2 + 2*px* + *q* = 0 differ by 2. Show that *p*2 = 1 + *q*
2. If the roots of the equation *ax*2 + *bx* + *c* = 0 are *α* and *β*, find expressions in terms of *a*, *b*, and *c* for:

(a) *α*2 *β* + *α β*2 (b) *α*2 + *β*2

(c) *α*3 + *β*3 (d) 

(e)  (f) *α*4 + *β*4

1. The equation *ax*2 + *bx* + *c* = 0 has roots *α* and *β*. Find equations whose roots are:

(a) −*α*, −*β* (b) *α* + 1, *β* + 1 (c) *α*2, *β*2

(d)  (e) *α* – *β*, *β* – *α* (f) 2 *α* + *β*, *α* + 2*β*

1. Prove that, if the difference between the roots of the equation *ax*2 + *bx* + *c* = 0 is 1, then *a*2 = *b*2 – 4*ac*
2. Prove that if one root of the equation *ax*2 + *bx* + *c* = 0 is twice the other, then 2*b*2 = 9*ac*
3. Prove that if the sum of the squares of the roots of the equation *ax*2 + *bx* + *c* = 0 is 1, then *b*2 = 2*ac* + *a*2.
4. Prove that if the sum of the reciprocals of the roots of the equation *ax*2 + *bx* + *c* = 0 is 1, then *b* + *c* = 0.

**Answers**

1. (a) 4 (b) 

2. (a)  (b)  (c) 3

3. (a) *x*2 + 6*x* + 3 (b) 3*x*2 – 6*x* + 1 = 0

(c) 25*x*2 – 14*x* + 1 = 0

4. 3*x*2 – 26*x* + 3 = 0

8. 

10. (*p* – *q* + *r*)*x*2 + 2(*r* – *p*)*x* + *p* + *q* + *r* = 0

13. , 4*x*2 – 13*x* + 1 = 0

14. *k* = 3, *α* = 3 17. *x* = 0, *x* = -2

18. *x* = -1, 

20. *p* = -2, *q* = -4 21. 2*x*2 – 3*x* – 4 = 0

22. 4*x*2 – 3*x* – 1. 23. *x*2 – 20*x* + 79.

24. *x*2 – 4*x* – 1 = 0

25. (i)  (ii) 4*x*2 + 11*x* + 9 = 0

29. *q*2*x*2 – (*p*2 – 2*q*)(*q*2 + 1)*x* + (*q*2 + 1)2 = 0

30. -11, 

31. *x*2 + 9*x* + 64 = 0

33. (i) (*x* + 3)2 + 7 (ii) 7 at *x* = -3

34. (a) , (b)  (c)  (d) 

35. (a) , (b)  (c)  (d) 

36. (a) 72, (b) 5 (c)  (d) -32

37. (a) *x*2 – 39*x* + 49 = 0

(b) *x*2 – 7*x* – 1 = 0

(c) *x*2 + 35*x* – 343 = 0

38. (a) 2*x*2 + 4*x* + 1 = 0

(b) *x*2 – 4*x* + 2 = 0

(c) *x*2 – 6*x* + 1 = 0

39. 4*x*2 – 49*x* + 36 = 0

40.  41. ±6

43. (a)  (b)  (c) 

(d)  (e)  (f) 

44 (a) *ax*2 – *bx* + *c* = 0

(b) *ax*2 + (*b* – 2*a*)*x* + *a* – *b* + *c* = 0

(c) *a*2*x*2 + (2*ac* – *b*2)*x* + *c*2 = 0

(d) *cx*2 – *bx* + *a* = 0

(e) *a*2*x*2 – (*b*2 – 4*ac*) = 0

(f) *a*2*x*2 + 3*abx* + (2*b*2 + *ac*) = 0

## POLYNOMIALS

A polynomial is an expression consisting of variables and co-efficient which only employs the operations of addition, multiplication and non-negative integer exponent.

Consider the expression

Where is said to be a polynomial of degree n. When we solve we get n unequal roots. When is equal to each of the unequal values

then

are factors of

Polynomials must have whole numbers as exponents for example is a polynomial but is not a polynomial.

Polynomials appear in a wide variety of areas of mathematics and science. For example they are used to form polynomial equations which encode a wide range of problems from elementary word problems to complicated problems in science. They are used in calculus to approximate other functions.

## Remainder Theorem

If when a polynomial is divided by, the quotient is and remainder is then;

If then is a factor of

This approach can be extended to the division of the polynomial by polynomial of the degree less or equal to the degree of .

If the division gives the quotient and remainder then

Where is of lower degree than

**The Remainder Theorem**

When is divided by, the remainder is

Proof

Equating the divisor to zero

**Example I**

Find the remainders when

1. 3*x*2 – 4*x*2 + 5*x* – 8 is divided by *x* – 2
2. 2*x*3 – 3*x*2 – 5*x* + 6 is divided by *x* + 2
3. 2*x*3 – 7*x* + 6 is divided by *x* – 3
4. *x*5 + *x* – 9 is divided by *x* + 1

***Solutions***

Where is the quotient, is the divisor and is the remainder

The remainder when is divided by

Equating the divisor to 0





The remainder when is divided by is -12

Equating the divisor to 0

*x* – 3 = 0

*x*  = 3

The remainder when is divided by is 39

Equating the divisor to 0

The remainder when is divided by is -11

|  |
| --- |
| **Suppose a polynomial *f*(*x*) has a repeated factor *x – a*. So that**  **So by differentiating**  ***(differentiation by product rule*)**  **Hence if has a repeated factor of then is also a factor of**  **If is a factor of a polynomial if and only if** |

**Example II**

Given that the polynomial *f*(*x*) = *x*3 + 3*x*2 – 9*x* + *k*  has a repeated linear factor, find the possible values of *k*.

***Solution***

The repeated factor of *f*(*x*) is either (*x* – 1) or (*x* + 3)

If is a factor of, then



If is a factor, then

The possible values of k are

**Obtaining the remainder by long division**

Here are the steps required for dividing by a polynomial containing more than one term

**Step I:** Make sure the polynomial is written in descending order. If any terms are missing, use a zero to fill in the missing term (this will help with the spacing)

**Step II:** Divide the term with the highest power inside the division symbol by the term with the highest power outside the division symbol

**Step III:** Multiply (or distribute) the answer obtained in the previous step by the polynomial in front of the division symbol

**Step IV:** Subtract and bring down the next term

**Step V:** Repeat step (II), (III) and (IV) until there are no more terms to bring down.

**Step VI:** Write the final answer. The term remaining after the last subtract step is the remainder and must be written as a fraction in the final answer.

**Example**

Find the remainder when is divided by

***Solution***

|  |  |
| --- | --- |
| STEP I  Make sure the polynomial is written in descending order. If any term is missing, use zero to fill in the missing terms (this will help with the spacing). In this case, the problem is ready as it is. | *x*3 – 4*x*2 + 2*x* – 3  *x* + 2 |
| STEP II  Divide the term with the highest power inside the division symbol by the term with the highest power outside the division symbol. In this case, we have *x*3 divided by which is *x*2 | *x*3 – 4*x*2 + 2*x* – 3  *x* + 2  *x*2 |
| STEP III  Multiply (or distribute) the answer obtained in the previous step by polynomial in front of division symbol. In this case, we need and *x* + 2 | *x*3 – 4*x*2 + 2*x* – 3  *x* + 2  *x*2  *x*3 + 2*x*2 |
| STEP IV  Subtract and bring down the next term | *x*3 – 4*x*2 + 2*x* – 3  *x* + 2  *x*2  *x*3 + 2*x*2  -6*x*2 + 2*x –* 3 |
| STEP V  Divide the term with the highest power inside the division symbol by the term with the highest power outside the division symbol. In this case, we have -6*x*2 divided by *x* which is -6*x*. | *x*3 – 4*x*2 + 2*x* – 3  *x* + 2  *x*2 – 6*x*  *x*3 – 2*x*2  -6*x*2 + 2*x* – 3 |
| STEP VI  Multiply (or distribute) the answer obtained in the previous step by the polynomial in front of division symbol. In this case, we need to multiply  (-6*x*) by *x*+2 | *x*3 – 4*x*2 + 2*x* – 3  *x* + 2  *x*2 – 6*x*  *x*3 – 2*x*2  -6*x*2 + 2*x*  -6*x*2 − 12*x* |
| STEP VII  Subtract and bring down the next term | *x*3 – 4*x*2 + 2*x* – 3  *x* + 2  *x*2 – 6*x*  *x*3 – 2*x*2  -6*x*2 + 2*x*  -6*x*2 − 12*x*  14*x* − 3 |
| STEP VIII  Divide the term with the highest power inside the division symbol by the term with the highest power outside the division symbol. in this case, we have divided by which is | *x*3 – 4*x*2 + 2*x* – 3  *x* + 2  *x*2 – 6*x* + 14  *x*3 – 2*x*2  -6*x*2 + 2*x*  -6*x*2 − 12*x*  14*x* − 3 |
| STEP IX  Multiply (or distribute) the answer obtained in the previous step by the polynomial in front of the division symbol. In this case, we need to multiply 14 by *x* + 2 | *x*3 – 4*x*2 + 2*x* – 3  *x* + 2  *x*2 – 6*x* + 14  *x*3 – 2*x*2  -6*x*2 + 2*x*  -6*x*2 − 12*x*  14*x* − 3  14*x* +28 |
| **STEP X**  Subtract and notice there are no more terms to bring down | *x*3 – 4*x*2 + 2*x* – 3  *x* + 2  *x*2 – 6*x* + 14  *x*3 – 2*x*2  -6*x*2 + 2*x*  -6*x*2 − 12*x*  14*x* − 3  14*x* +28  -31 |
| **STEP XI**  Write the final answer. The term remaining after the last subtract step is the remainder and must be written as a fraction. | *x*2 – 6*x* + 14 + |

The remainder when *x*3 – 4*x*2 + 2*x* – 3 is divided by *x* + 2 is -31.

**Example II**

By using long division, obtain remainders and quotients when

1. is divided by
2. is divided by
3. is divided by
4. is divided by
5. is divided by

***Solution***

1. *x*3 + 3*x*2 – 4*x* – 12 is divided by *x*2 + *x* – 6

*x*3 + 3*x*2 – 4*x* – 12

*x*2+*x*-6

*x* + 2

*x*3 + *x*2 − 6*x*

2*x*2 + 2*x* – 12

2*x*2 + 2*x* – 12

0



*R* = 0

*Q*(*x*) = *x* + 2(the quotient)

1. **2*x*4 – 8*x*3 + 5*x*2 + 4** is divided by *x* – 3

2*x*4 – 8*x*3 + 5*x*2 +0*x* + 4

2*x*4 – 8*x*3 + 5*x*2 +0*x* + 4

*x* − 3

2*x*3 – 2*x*2 – *x* – 3

2*x*4 – 6*x*3

-2*x*3 + 5*x*2 + 0*x* + 4

-2*x*3 + 6*x*2

-*x*2 + 0*x* + 4

-*x*2 + 3*x*

-3*x* + 4

-3*x* + 9

-5

Where Quotient and remainder = *R*

1. 5*x*3 – 6*x*2 + 3*x* + 14

5*x*3 – 6*x*2 + 3*x* + 14

*x* + 1

5*x*2 – 11*x* + 14

5*x*3 + 5*x*2

-11*x*2 + 3*x* + 14

-11*x*2 − 11*x*

14*x* + 14

14*x* + 14

0

1. 2*x*4 + 6*x*3 − 7*x*2 +9*x* + 11

2*x*4 + 6*x*3 − 7*x*2 +9*x* + 11

*x +* 4

2*x*3 – 2*x*2 + *x* + 5

2*x*4 + 8*x*3

-2*x*3 − 7*x*2 + 9*x* + 11

-2*x*3 − 8*x*2

*x*2 + 9*x* + 11

*x*2 + 4*x*

5*x* + 11

5*x* + 20

-9



1. *x*4 – 16 is divided by *x* – 2

*x*4 + 0*x*3 + 0*x*2 +0*x* − 16

*x −* 2

*x*3 +2*x*2 + 4*x* + 8

*x*4 − 2*x*3

2*x*3 + 0*x*2 + 0*x* − 16

2*x*3 − 4*x*2

4*x*2 + 0*x* − 16

4*x*2 − 8*x*

8*x* − 16

8*x* − 16

0

Were

And

**Obtaining the remainder by synthetic approach**

**Definitions** :

Dividend: The number or expression you are dividing into

Divisor: The number or expression you are dividing by

Synthetic division: is a quick method of dividing a polynomial when the divisor is of the form

**Steps involved when obtaining the remainder by synthetic approach**

1. Write the value obtained after equating the divisor to 0 and the coefficients of the dividend in descending order in the first row. If any terms are missing, place a zero in its place
2. Bring the leading coefficient in the top row down to bottom (third) row
3. Next multiply the number in the bottom row by *c* and place this product in the second row under the next coefficient and add these two terms together
4. Continue with this process until you reach the last column
5. The numbers in the bottom are coefficients of the quotient and the remainder. The quotient will have one degree less than the dividend

**Example I**

Use synthetic approach to obtain the remainder when is divided by

***Solution***

First note that the term is missing so we must record zero in its place

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *x*4 | *x*3 | *x*2 | *x* | *x*0 |
| 2 | -8 | 5 | 0 | 4 |
|  | 6 | -6 | -3 | -9 |
| 2 | -2 | -1 | -3 | -5 |

*x* = 3

Therefore, the quotient is and the remainder is -5.

**Example II**

Use synthetic approach to obtain the remainders when

1. is divided by
2. is divided by
3. is divided by

***Solution***

1. Equating the divisor to zero

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *x*4 | *x*3 | *x*2 | *x* | *x*0 |
| 1 | 0 | 0 | 0 | 16 |
|  | -1 | 1 | -1 | 1 |
| 1 | -1 | 1 | -1 | -15 |

*x* = -1

Where remainder

And Quotient

|  |
| --- |
| **Note: Synthetic method only work where the divisor is of the form** |

1. 5*x*3 – 6*x*2 + 3*x* + 14

Equating the divisor to zero

*x +* 1 = 0, *x*  = -1

|  |  |  |  |
| --- | --- | --- | --- |
| *x*3 | *x*2 | *x* | *x*0 |
| 5 | -6 | 3 | 14 |
|  | -5 | 11 | -14 |
| 5 | -11 | 14 | 0 |

*x* = -1

*R =* 0

Where

*R* is the remainder

*D*(*x*) = *x* + 4

Equating the divisor to zero

*x* + 4 = 0

*x* = -4

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *x*4 | *x*3 | *x*2 | *x* | *x*0 |
| 2 | 6 | -7 | 9 | 11 |
|  | -8 | 8 | -4 | -20 |
| 2 | -2 | 1 | 5 | -9 |

*x* = -4

*Q*(*x*) = 2*x*3 – 2*x*2 + *x* + 5

*R = -*9

Where *Q*(*x*) = quotient, *R* = remainder

**More examples on polynomials**

Find the values in the expression below if the following conditions are satisfied

1. has a remainder -3 when divided by
2. has a remainder 8 when divided by

***Solution***

…… (1)

Equating the divisor to zero

Substituting *x* = 2 in Eqn (1);

1. …. (1)

Substituting *x* = 2 in Eqn (1);

**Example II**

Show that is divisible by and find other factors

***Solution***

12*x*3 + 16*x*2 − 5*x* − 3

2*x* − 1

6*x*2 + 11*x* + 3

12*x*3 − 6*x*2

22*x*2 − 5*x* − 3

22*x*2 − 11*x*

6*x* − 3

6*x* − 3

Since the remainder is zero

is divisible by



For

The other Factors are 2, 9 and product of factors is 18



6*x*2 + 11*x* + 3 = (2*x* + 3)(3*x* + 1)

The other factors of are and

**Example III**

is a factor of . Find the remainder when the expression is divided by

***Solution***

Since is a factor,

*x* + 2 = 0  *x* = -2

(Since *x* + 2 is a factor of *P*(*x*)

(Remainder = 0)

**Example** **IV**

The remainder obtained when is divided by is twice the remainder obtained when the same expression is divided by. Find

***Solution***



P(3)

**Example V**

A cubic polynomial has a remainder 72 when divided by and exactly divisible by. Calculate the values of. Show that is also a factor. Obtain the other factor factor

***Solution***:

………………………….. (1)

………………………… (2)

Eqn. (1) – eqn. (2)

-3 = 3*a*

*a* = -1

Substituting *a* = -1 in Eqn (2);

 -1 = 1 + *b*

*b* = -2

P(*x*) = 6*x*3 + 7*x*2 – *x* – 2

We can apply long division to obtain other factors

6*x*3 + 7*x*2 − *x* − 2

*x* + 1

6*x*2 + *x* − 2

6*x*3 + 6*x*2

*x*2 − *x* − 2

*x*2 + *x*

-2*x* − 2

-2*x* − 2

(6*x*2 + *x* – 2)(*x* + 1)

But 6*x*2 + *x* – 2 = 6*x*2 + 4*x* – 3*x* – 2

= 2*x*(3*x* + 2) – 1(3*x* + 2)

6*x*3 + 7*x*2 – *x* – 2 = (2*x* – 1)(3*x* + 2)(*x* + 1)

The other factors are (3*x* + 2) and (*x* + 1)

**Example VI**

and are factors of and it leaves a remainder of 12 when divided by .find the values of *a*, *b*, and *c*.

***Solution***

…………………… (1)

……………………… (2)

………………….. (3)

Eqn. (1) – eqn.(2)

Eqn.(1) – eqn.(3)

But *b* = -1

Substituting in eqn. (1)

**Example IX**

When a polynomial is divided by , the remainder is 4 and when is divided by , the remainder is 7. Find the remainder when is divided by

***Solution***

P(2) = 4 and P(3) = 7

The remainder is of the form

Equating the divisor to zero

But

…………………………… (1)

…………………………… (2)

Eqn (2) – Eqn (1)

*a =* 3

Substituting in eqn. (1)

*R*(*x*) = *ax + b*

**Example VIII**

When a polynomial is divided by, the remainder is 5 and when is divided by, the remainder is 7. Find the remainder when the same expression is divided by.

***Solution***

The remainder takes the form *R*(*x*) = *ax* + *b*.

Equating the divisor to zero

…………………………… (1)

………………………….. (2)

Eqn. (2) – eqn. (1)

Substituting in eqn. (1)

The remainder is

**Example IX**

Given that the polynomial where is a quotient, and is a remainder. Show that

***Solution***

*f*(*x*) = *Q*(*x*)*D*(*x*) + *R*(*x*)

Where

Equating the divisor to zero

……………………… (1)

……………………… (2)

Eqn. (1) – eqn. (2)

Substituting in equation (1)

But since

**Example**

If a factor of . Find the values of p and q

***Solution***

******

But







Equating co-efficients of the same monomial;



*p =* 1 and *q* = -7

**Example XII**

If *f*(*x*)and *g*(*x*) are polynomials.

. Find and hence find A and B in terms of and deduce that is a repeated factor of if and only if

***Solution***



If is a repeated factor of

If (*x* – *a*)2 is a repeated factor of *f*(*x*) if and only if

**Example (UNEB 2015)**

**(a)** Given that *f*(*x*) = (*x* – α)2*g*(*x*), show that *f '*(*x*) is divisible by (*x* – α)

**(b)** A polynomial *P*(*x*) = *x*3 + 4*ax*2 + *bx* + 3 is divisible by (*x* – 1)2. Use your results above to find the values of *a* and *b*. Hence solve the equation *p*(*x*) = 0

***Solution***

*f*(*x*) = (*x* – α)2*g*(*x*)

*f* '(*x*) = (*x* – α)2*g*'(*x*) + *g*(*x*)2(*x* – α)

* f* '(*x*) is divisible by (*x* – α)

*p*(*x*) = *x*3 + 4*ax*2 + *bx* + 3

*p*'(*x*) = 3*x*2 + 8*ax* + *b*

Since *x* – 1 is a factor of *p*(*x*) and *p*'(*x*),

*p*(1) = 0 and *p*'(1) = 0

1 + 4*a* + *b* + 3 = 0

4*a* + *b* = -4 …………………. (i)

*p*'(1) = 0

3 + 8*a* + *b* = 0

8*a* + *b* = -3 …………………. (ii)

Eqn (ii) – Eqn (i);

4*a* = 1



Substituting ** in Eqn (ii)



2 + *b* = -3

*b* = -5

*p*(*x*) = *x*3 + *x*2 + -5*x* + 3

*p*(1) = 1 + 1 + 3 – 5 = 0

*p*'(*x*) = 3*x*2 + 2*x* – 5

*p*'(1) = 3 + 2 – 5

*p*'(1) = 0

*P*(*x*) = *x*3 + *x*2 – 5*x* + 3

(*x*3 *+ x*2 *–* 5*x +* 3) *=* (*x* – 1)2*g*(*x*)

(*x*3 + *x*2 – 5*x* + 3) = (*x*2 – 2*x* + 1)*g*(*x*)

*x*3 + *x*2 − 5*x* + 3

*x*2– 2*x* + 1

*x* + 3

3*x*2 − 6*x* + 3

*x*3 − 2*x*2 + *x*

3*x*2 − 6*x* + 3

(*x*3 + *x*2 – 5*x* + 3) = (*x*2 – 2*x* + 1)(*x* + 3)

(*x*2 – 2*x* + 1)(*x* + 3) = 0

(*x* – 1)2(*x* + 3) = 0

*x* – 1 = 0 OR *x* + 3 = 0

*x* = 1  *x* = -3

**Example (UNEB Question)**

When the quadratic expression *ap*2 + *bp* + *c* is divided by *p* − 1, *p* − 2 and *p* + 1, the remainders are 1, 1 and 25 respectively. Determine the factors of the expression.

**b**) Express 2*x*3 + 5*x*2 − 4*x* − 3 in the form

(*x*2 + *x* − 2) *Q*(*x*) + *Ax + B*; where *Q*(*x*) is a polynomial in *x* and *A* and *B* are constants. Determine the values of A and *B* and the expression *Q*(*x*).

***Solution***

Let *f*(*p*) = *ap*2 + *bp + c*

Now *f*(1) = *a* + *b* + *c*

But *f*(1) = 1

⟹ *a* + *b* + *c* = 1 ……………… (i)

*f*(2) = 4*a* + 2*b* + *c*

But *f*(2) = 1

⟹ 4*a* + 2*b* + *c* = 1…………….. (ii)

*f*(-1) = *a* – *b* + *c*

But *f*(-1) = 25

=> *a* – *b* + *c* = 25 …………….. (iii)

Eqn (i) – Eqn (ii)

-3*a* − 3*b* = 0

-3*a* = *b*…………………….(iv)

Eqn (ii) – Eqn (iii)

3*a* + 3*b* = -24

*a* + *b* = -8 ………………….(v)

*a* – 3*a* = -8

-2*a* = -8

*a* = 4

Substituting for a into Eqn (iv)

*b* = -12

Substituting for and b into Eqn (i)

4 – 12 + *c* = 1

-8 + *c* = 1

*c* = 9

Hence *f*(*p*) = 4*p*2 – 12*p* + 9

By factorization,

4*p*2 – 12*p* + 9 = 4*p*2 – 6*p* – 6*p* + 9

= 2*p*(2*p* – 3) – 3(2*p* – 3)

= (2*p* – 3)(2*p* – 3)

Hence the factors of 4*p*2 – 12*p* + 9 are

(2*p* – 3) and (2*p* – 3)

**b**) Let 2*x*3 + 5*x*2 – 4*x* – 3

≡ (*x*2 + *x* – 2)(2*x* + D) + *Ax* + *B*

By opening brackets on the L.H.S

2*x*3 + 5*x*2 – 4*x* – 3 ≡ 2*x*3 + *Dx*2 + 2*x*2 + *Dx* – 4*x* – 2*D* + *Ax* + *B*

2*x*3 + 5*x*2 – 4*x* – 3 ≡ 2*x*3 +(*D* + 2)*x*2 + (*D* – 4)*x* – 2*D* + *Ax* + *B*

2*x*3 + 5*x*2 – 4*x* – 3 ≡ 2*x*3 + (*D* +2)2 + (*D* + *A* – 4)*x* – 2*D* + *B*

Equating corresponding coefficients,

For *x*2,

*D* + 2 = 5

*D* = 3

For *x*

-4 = *D* + *A* – 4

*D* = -*A*

3 = -*A*

*A* = -3

For constant

-3 = -6 + *B*

*B* = 6 – 3

*B* = 3

Hence 2*x*3 + 5*x*2 – 4*x* – 3 ≡ (*x*2 + *x* – 2) (2*x* +3 ) – 3*x* + 3

***Alternatively***

By using long division,



Hence 2*x*3 + 5*x*2 – 4*x* – 3 ≡ (*x*2 + *x* – 2) (2*x* +3 ) – 3*x* + 3

## Revision Exercise

1. Find the constants *p*, *q* and *r* such that

2*y*2 – 9*y* + 14 = *p*(*y* – 1)(*y* – 2) + *q*(*y* – 1) + *r*

1. Find the relationship between *p* and *r* so that

*A*2 + 3*qA*2 + *PA* + *R* shall be a perfect cube for all values of *A*.

1. When the expression *p*6 + 4*p*2 + *ap* + *b* is divided by *p*2 – 1, the remainder is 2*p* + 3. Find the values of *a* and *b*.
2. Find the remainder when:

(a) 4*x*3 – 5*x* + 4 is divided by –(1 – 2*x*)

(b) *y*5 + *y* – 9 is divided by *y* + 1

1. Find the values of *β* in the expressions below when the following conditions are satisfied:

(a) *y*3 + *βy*2 + 3*y* – 5 has remainder -3 when divided by *y* – 2.

(b) *x*5 + 4*x*4 – 6*x*2 – *βx* + 2 has a remainder 6 when divided by .

1. (*p* – 1) and (*p* + 1) are factors of the expression

*p*3 + *ap*2 + *bp* + *c* and it leaves a remainder of 12 when divided by *p* – 2. Find the values of *a*, *b*, *c*.

1. The expression *ax*4 + *bx*3 + 3*x*2 – 2*x* + 3 has a remainder *x* + 1 when divided by *x*2 – 3*x* + 2. Find the values of *a* and *b*.
2. What is the value of *a* if 2*x*2 – *x* – 6, 3*x*2 – 8*x* + 4 and *ax*3 – 10*x* – 4 have a common factor?
3. Factorise the expression 3*k*3 – 11*k*2 – 19*k* – 5.
4. Find the values of *a* and *b* which make

*y*4 + 6*y*3 + 13*y*2 + *ay* + *b* a perfect square.

1. If *x*2 + *nx* + q and *x*2 + *dx* + *m* have a common factor (*x* – *p*). show that .
2. The remainder obtained when 2*x*3 + *ax*2 – 6*x* + 1 is divided by (*x* + 2) is twice the remainder obtained when the same expression is divided by (*x* – 1). Find the values of *a* and *b*.
3. Given that (*x* + 2) is a factor of 2*x*3 + 6*x*2 *+ bx* – 5, find the remainder when the expression is divided by (2*x* – 1)
4. Find the values of *p* and *q* if the expression 2*y*3 – 15*y*2 + *py* + *q* is divisible both by *y* – 4 and 2*y* – 1.
5. Use the remainder theorem to find the factors of

*x*4 + 3*x*2 – 4.

1. Find *p* and *q* so that

*y*4 – 7*y*3 + 17*y*2 – 17*y* + 6 = (*y* – 1)2(*y*2 + *py* + *q*) Hence find all the factors of the quadratic equation.

1. Factorise (a) 2*y*3 – *y*2 + 2*y* – 1

(b) 2*y*3 + 5*y*2 + *y* – 2

1. Use the synthetic approach to find the remainder when:

(a) 8*y*3 – 10*y*2 + 7*y* + 3 is divided by 2*y* – 1

(b) 5 + 6*x* + 7*x*2 – *x*3 is divided by *x* + 2

1. Find the range of values of *q* for which

(2 – 3*q*)*x*2 + (4 – *q*)*x* + 2 = 0 has no real roots.

1. Find the value of *k* for which the line *y = mx + c* is a tangent to the curve *x*2 + *xy* + 2 = 0.
2. Express the polynomial *f*(*x*) = 2*y*4 + *y*3 – *y*2 + 8*y* – 4 as a product of two linear factors and a quadratic factor *q*(*y*). Prove that there are no real values of *y* for which *q*(*y*) = 0.
3. The polynomial *ax*3 + *bx*2 – 5*x* + 1 has 2*x* – 1 and *x* – 1 as its two factors. Find *a* and *b*.
4. *f*(*x*) = 2*x*3 + *px*2 + *qx* + 6 where *p* and *q* are constants. When f(x) is divided by *x* – 1, the remainder is -6, when divided by (*x* + 1) the remainder is 12. Show that *f*(½) = 0 hence write *f*(*x*) as a product of linear factors.
5. Find the remainder when

(a) 3*x*5 – *x*2 + 1 is divided by *x* + 2

(b) *x*4 – 2*x*2 + 3*x* – 6 is divided by *x*2 + 4*x* + 3

1. Use long division to find the missing factors:

(a) *x*5 + *x*4 + 3*x*3 + 5*x*2 + 2*x* + 8 = (*x*2 – *x* + 2)(…)

(b) 6*x*5 + *x*4 – *x*3 – 15*x* + 5 = (3*x* – …)(….)

1. The expression 2*x*3 + *ax*2 + *bx* + 6 is exactly divisible by (*x* – 2) and on division by (*x* + 2) gives a remainder of -12. Calculate the values of *a* and *b* and factorise the expression completely.
2. *f*(*x*) = *x*2 + *ax* + *b* when *f*(*x*) is divided by *x* – 2 the remainder is 8 and when *f*(*x*) is divided by *x* + 3 the remainder is 18. Find the values of constants *a* and *b*.
3. If *f*(*x*) denotes the polynomial 2*x*3 – 3*x*2 – 8*x* – 3, find the remainders when *f*(*x*) is divided by:
4. *x* – 1 (ii) *x +* 3 (iii) 2*x* + 1
5. State the remainder when the cubic polynomial *x*3 + *ax*2 – 3*x* + 4 is divided by (*x* – 3) the remainder obtained is twice the remainder obtained when the polynomial is divided by (*x* – 2). Calculate *a*.
6. When *f*(*x*) = *x*4 – 2*x*3 + *ax*2 – *bx* + *c* is divided by *x* – 2, the remainder is -24 and when divided by *x* + 4, the remainder is 240. Given that *x* + 1 is a factor of *f*(*x*), show that *x* – 1 is also a factor.
7. Given that *f*(*x*) = *x*3 + *kx*2 – 2*x* + 1, When *f*(*x*) is divided by (*x* – *k*), the remainder is *k*. Find the possible values of *k*.
8. When the polynomial *p*(*x*) is divided by (*x* – 1), the remainder is 5 and when p(*x*) is divided by (*x* – 2), the remainder is 7. Find the remainder when *p*(*x*) is divided by (*x* – 1)(*x* – 2).
9. When the polynomial *p*(*x*) is divided by (*x* – 2), the remainder is 4 and when *p*(*x*) is divided by (*x* – 3) the remainder is 7. Find by writing *p*(*x*) = (*x* – 2)(*x* – 3)*q*(*x*) + *ax*, the remainder when *p*(*x*) is divided by (*x* – 2)(*x* – 3). If *p*(*x*) is cubic when the coefficient of *x*3 is unity and *p*(1) = 1 determine *q*(*x*).
10. Find the quotient and remainder when:

(a) 6*x*2 – *x* + 2 is divided by 2*x* + 1

(b) 6*x*2 – 7*x* + 5 is divided by 2*x* – 3

(c) *x*3 + 3*x*2 – 2*x* + 1 is divided by *x* – 2

(d) 2*x*3 – 3*x*2 – 4*x* + 1 is divided by *x* – 4

(e) 4*x*2 – 3*x*2 + *x* + 2 is divided by 2*x* + 3

1. Use the remainder theorem to find the remainder when

(a) 3*x*2 + 2*x* – 4 is divided by *x* – 2

(b) 2*x*3 + 4*x*2 – 6*x* + 5 is divided by *x* – 1

(c) 8*x*3 + 4*x* + 3 is divided by 2*x* – 1

(d) 6*x*3 – 2*x*2 + 5*x* – 4 is divided by *x*

(e) 3*x*3 + 6*x* – 8 is divided by *x* + 3

1. The expression 2*x*3 – 3*x*2 + *ax* – 5 gives a remainder of 7 when divided by *x* – 2. Find the value of the constant *a*.
2. The remainder when *x*3 – 2*x*2 + *ax* + 5 is divided by (*x* – 3) is twice the remainder when the same expression is divided by *x* + 1. Find the value of the constant *a*.
3. The remainder when *cx*3 + 2*x*2 – 5*x* + 7 is divided by *x* – 2 is equal to the remainder when the same expression is divided by *x* + 1. Find the value of the constant *c*.
4. Given that *x* – 4 is a factor of 2*x*3 – 3*x*2 – 7*x* + *b*, where *b* is a constant. Find the remainder when the same expression is divided by 2*x* – 1.
5. The expression *cx*3 + *dx*2 + 3*x* + 8 leaves a remainder of -6 when divided by *x* – 2 and a remainder of -34 when divided by *x* + 2. Find the value of the constants *c* and *d*.
6. The expression *x*3 – *x*2 + *ax* + b has a factor of *x* + 3, and leaves a remainder of 6 when divided by *x* – 3. Find the values of the constants *a* and *b* and hence factorise the expression.
7. The remainder when the expression *x*3 – 2*x*2 + *ax* + *b* is divided by *x* – 2 is five times the remainder when the same expression is divided by *x* – 1, and 12 less than the remainder when the same expression is divided by *x* – 3. Find the values of constants *a* and *b*.
8. Show that (*x* – 2) is a factor of *x*3 – 9*x*2 + 26*x* – 24. Find the set of values of *x* for which *x*3 – 9*x*2 + 26*x* – 24 < 0
9. The expression 6*x*2 + *x* + 7 leaves the same remainder when divided by *x* – *a* and by *x* + 2*a*, where *a* ≠ 0. Calculate the value of *a*.
10. Given that *x*2 + *px* + *q* and 3*x*2 + *q* have a common factor *x* – b, where *p*, *q* and *b* are non-zero. Show that 3*p*2 + 4*q* = 0.
11. Express the polynomial *f*(*x*) = 2*x*4 + *x*3 – *x*2 + 8*x* – 4 as a product of two linear factors and a quadratic factor *q*(*x*). Prove that there are no real values of *x* for which *q*(*x*) = 0.
12. Find the remainder when:
13. *x*3 + 3*x*2 – 4*x* + 2 is divided by *x* – 1
14. *x*3 – 2*x*2 + 5*x* + 8 is divided by *x* – 2
15. *x*5 + *x* – 9 is divided by *x* + 1
16. *x*3 + 3*x*2 + 3*x* + 1 is divided by *x* + 2
17. Find the values of *a* in the expressions below when the following conditions are satisfied.
18. *x*3 + *ax*2 + 3*x* – 5 has remainder -3 when divided by *x* – 2.
19. *x*3 + *x*2 + a*x* + 8 is divisible by *x* – 1
20. *x*3 + *x*2 – 2*ax* + *a*2 has remainder 8 when divided by *x* – 2
21. *x*4 – 3*x*2 + 2*x* + a is divisible by *x* + 1
22. Show that 2*x*3 + *x*2 – 13*x* + 6 is divisible by *x* – 2, and hence find the other factors of the expression.
23. Show that 12*x*3 + 16*x*2 – 5*x* – 3 is divisible by 2*x* – 1 and find the factors of the expression.
24. Factorise:

*x*3 – 2*x*2 – 5*x* + 6

*x*3 – 4*x*2 + *x* + 6

2*x*3 + *x*2 – 8*x* – 4

2*x*3 + 5*x*2 + *x* – 2

1. Find the values of *a* and *b* if *ax*4 + *bx*3 – 8*x*2 + 6 has remainder 2*x* + 1 when divided by *x*2 – 1
2. The expression *px*4 + *qx*3 + 3*x*2 – 2*x* + 3 has remainder *x* + 1 when divided by *x*2 – 3*x* + 2. Find the values of *p* and *q*.
3. The expression *ax*2 + *bx* + *c* is divisible by *x* – 1, has remainder 2 when divided by *x* + 1 and has remainder 8 when divided by *x* – 2. Find the values of *a*, *b* and *c*.
4. (*x* – 1) and (*x* + 1) are factors of the expression

*x*3 + *ax*2 + *bx* + *c* and leaves a remainder of 12 when divided by *x* – 2. Find the values of *a*, *b*, and *c*.

1. What are the values of *a* and *b* if *x* – 3 and *x* + 7 are factors of the quadratic equation *ax*2 + 12*x* + *b*?
2. Show that 3*x*3 + *x*2 – 8*x* + 4 is zero when *x* = 2/3, hence factorise the expression.

**Answers**

1. *p* = 2, *q* = -3, *r* = 7 2. *P*3 = 27*R*2

3. *a* = 1, *b* = -1 4. (a) 2 (b) -11

5. (a) -3 (b) -2 6. *a* = 2, *b* = -1, *c* = -2

7. *a* = 1, *b* = -3 8. 3

9. (*k* + 1)(*x* – 3)(3*k* + 1) 10. *a* = 12, *b* = 4

12. 1½ 13. -2.5

14. *p* = 31, *q* = -12 15. (*x* + 1)(*x* – 1)(*x* – 2)

16. *p* = -5, *q* = 6; (*y* – 1)(*y* – 2)(*y* – 3)

17. (a) (*y*2 + 1)(2*y* – 1) (b) (*y* + 1)(*y* + 2)(2*y* – 1)

18. (a) 5 (b) 29 19. -16 < *q* < 0

20. ±4 21. (2*y* – 1)(*y* – 2)(*y*2 – *y* – 2)

22. *a* = 2, *b* =1

23.(a) *p* = 3, *q* = -11 b) (2*x* – 1)(*x* + 2)(*x* – 1)

24. (a) -99 (b) 35*x* – 39

25. (a) *x*3 + 2*x*2 + 3*x* + 4 = 0 (b) 2*x*4 + *x*3 – 5

26. *a* = -9, *b* = 7, [(*x* – 1)(*x* – 3)(2*x* + 1)]

27. *a* = -1, *b* = 6 28. (i) -12 (ii) -60 (iii) 0

29. *a* = -10 30. *a* = -9, *b* = 2, *c* = 8

31. *k* = 1, ½ ±  32. (2*x* + 3)

33. [3*x* – 2, (*x* – 1)]

34.(a) 3*x* – 2, 4 (b) 3*x* + 1, 8

(c) *x*2 + 5*x* + 8, 17 (d) 2*x*2 + 5*x* + 16, 65

(e) 2*x*2 – 3*x*2 + 3*x* – 4, 14

35. (a) 24 (b) 5 (c) 6 (d) -4 (e) 1

36. *a* = 4 37. *a* = -2

38. *c* = 1 39. *b* = 52, remainder = -56

40. *c* = 1, *d* = -7 41. -8, 12, (*x* – 1)2(*x* + 3)

42. 3, -1 43. *x* < 2 or 3 < *x* < 4

44.  46. (2*x* – 1)(*x* + 2)(*x*2 – *x* + 2)

47. (a) 2 (b) 18 (c) -11 (d) -1

48. (a) -3 (b) -10 (c) 2 (d) 4

49. (*x* + 3)(2*x* – 1) 50. (2*x* – 1)(2*x* + 3)(3*x* + 1)

51. (a) (*x* – 1)(*x* + 2)(*x* – 3) (b) (*x* + 1)(*x* – 2)(*x* – 3)

(c) (2*x* + 1)(*x* – 2)(*x* + 2) (d) *x* + 1)(*x* + 2)(2*x* – 1)

52. *a* = 3, *b* = 2 53. *p* = 1, *q* = -3

54. *a* = 3, *b* = -1, *c* = -2 55. *a* = 2, *b* = -1, *c* = -2

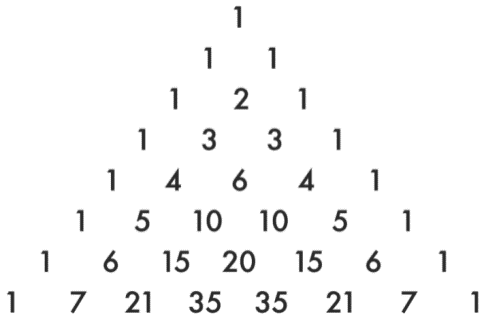
56. *a* = 3, *b* = -63 57. (*x* – 1)(*x* + 2)(3*x* – 2)

# BINOMIAL THEOREM

**Objectives of the topic:**

* Create rows of Pascal’s triangle
* Compute factorial values
* Compute binomial co-efficient by the formula
* Expand powers of binomial by Pascal’s triangle and by binomial theorem
* Approximate numbers using binomial expansion

**Pascal’s Triangle**



We can use Pascal triangle to expand expressions of the form (*a* + *b*)*n*

Pascal’s triangle helps us to calculate the powers of a binomial (*a* + *b*)*n* without actually multiplying it

**Note**

The literal factors are all combination of *a* and *b* where the sum of the components of the power is 4

The degree of each term is 4. The first term is actually which is *a*4(1).

Thus, to expand we would anticipate the following terms in which the sum of all the components of the powers is 5.

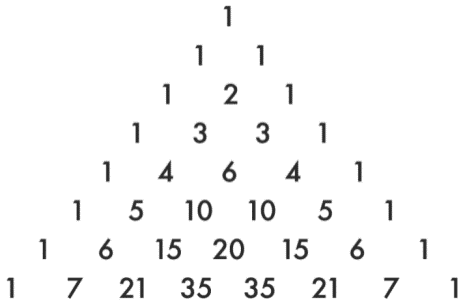
The question is what are the co-efficients. We can obtain the co-efficients from the Pascal’s triangle above (line five above)

**Example**

Use Pascal’s triangle to expand the following.

***Solution***

Consider the Pascal’s triangle



We can use Pascal’s triangle to find (*x* + 2)5

We can obtain the coefficients from the Pascal’s triangle above (line 6).

**(b)** (2*x* – 3)3 = (2*x*)3 + 3(2*x*)2(-3) + 3(2*x*)(-3)2 + (-3)3



(c) ****





= 64 – 576*x* + 2160*x*2 – 4320*x*3 + 4860*x*4 + -2916*x*5

+ 729*x*6

We have that Pascal’s triangle can be used to expand (*a* + *b*)*n* for the known value of n where n is a positive integer

However, as n becomes large it becomes difficult to determine the co-efficient of a triangle. Imagine a task to expand (*a* + *b*)10000.

This is so tedious yet indeed, we may not require all the terms of the expansion but just fewin the above case, we can use binomial theorem.

**Binomial Theorem**

It states that if n is a positive integer then

It is also stated as

An important particular case is when and giving

The binomial expansion discussed up to now is for the case when the exponent is positive

For the case when the number n is not positive, the binomial expansion (1 + *x*)*n* is valid when -1 < *x* < 1 OR |*x*| < 1.

**Example**

Expand

***Solution***

Using the binomial theorem;





**Example II**

Expand (2*x* – 3*y*)4

***Solution***

Using binomial expansion,





Example III

Expand

***Solution***

******

The *r***th** term in a binomial expansion

(co-efficient of a term in a binomial expansion)

The *r*th term of a binomial expansion is given by



|  |
| --- |
|  |

**Example** **I**

Write down the terms indicated in the expansion of the following and simplify your answers.

***Solution***

(*x* + 2)8 = (*a* + *b*)*n*

*n* = 8, *a* = *x*, *b* = 2

**(a)**

**(b)**

(3*x* – 2)5 = (*a* + *b*)*n*

*a* = 3*x*, *b* = -2



5 – *r* = 3

*r* = 2



**(c)**



**(d)**

= 11*Cr* 211 – *r x*11 – *ry r*

**Example III**

Find the term independent of x in expansion of

***Solution***



**Example IV**

Find the co-efficient of *x* in the expansion of

***Solution***

, *n* = 10

 10 – 3*r* = 1

= 10*C*323*x*

= 960*x*

The coefficient is 960

**Example V**

Find the co-efficient of the term in in the expansion

***Solution***



The coefficient is -16

## Validity of a Binomial Expansion



When n is not positive the binomial theorem is valid for or when

**Example I**

State what values of *x* for which the following expansions are valid:

***Solution***

is valid when |*x*|<1

So is valid

**(b)**

is valid when 

**(c)**

**(d)**

|*x*| > 1 for expansion

**More examples on Binomial Expansion**

**Example** **I (UNEB Question)**

as far as the term in *x*3. Hence evaluate

***Solution***

******

******

Comparing

0.961506545

= 3(0.961506545)

= 2.8845

**Example II**

Determine the expansion of as far as the term in Hence evaluate

***Solution***

******

Comparing



**Example III**

Expand up to and including the term in *x*3.***Solution***



**Example IV**

Expand as far as the term in.

***Solution***



**Example V**

Expand  as far as the fourth term

***Solution***

******

**Example** V**I**

Expand  as far as the third term

***Solution***

****



But

**Example VII**

Find the expansion of  as far as the term *x*−3

***Solution***:

******



**Example VIII**

Expand (1 – *x*)3(2 + *x*)6 up to the term in *x*2.

***Solution***

(1 – *x*)3(2 + *x*)6



= 64 + 192*x* + 240*x*2 – 192*x* – 576*x*2 + 192*x*2+ …

= 64 – 144*x*2

**Example IX**

Expand  up to and including the term in *x*2.

***Solution***









**Example** **X**

Show that, if *x* is small enough for its cube and higher powers to be neglected,  


By putting , show that ***Solution***



From

By putting



+ …

**Example XI**

Use Binomial expansion to expand  up to and including the term in *x*3. Hence find to 4 decimal places. Hence deduce the square root of 51

***Solution***



= 1 + 2*x*2 + 2*x* + 4*x*3 + …

**Example XII**

Given that, the first three terms of the expansion in ascending powers of *x* of (1 + *x* + *x*2)*n* are the same as the first three terms in the expansion of , find the values of *a* and *n*.  
***Solution***

from

(1 + *x* + *x*2)*n* = 

…… (1)



Comparing Eqn (1) and Eqn (2);

*n* = 12*a*



**Example XIII**

Find the first three terms of the expansion  
 in ascending powers of *x*. Deduce the approximate value of 

***Solution***







**Example XIV**

Write down the expansions in ascending powers of *x* up to the term in *x*2.   
**(a) (b)** Hence or otherwise Expand in ascending power of *x* up to the term in *x*2. By substituting , use your expansion to find the .

***Solution***

…

+…



When

**Example (UNEB Question)**

(a) By using the binomial theorem, expand  as far as the 4th term. Hence evaluate to one decimal place.

b) Find the coefficient of *x* in the expansion of 

***Solution***

a) By using binomial distribution theorem,



Now 

Here *n* =  and *y = –*3*x*

By substitution,

Evaluating :



Comparing  with 



b) When expanding, we could use Pascal’s triangle to get the coefficient of *x*, but this may seem tedious, so by using binomial expansion



Here, *y* =  *n* = 10





The coefficient of *x* is 960

**OR** By using the expansion of

The coefficient of *x* is 960

**OR** We could handle it by using direct approach

The term in is the one needed which is expanded as,



The coefficient = 

= 120 × 8

= 960

# Revision Exercise

1. Write down and simplify the terms indicated in the expression of the following in ascending powers of *x.*
2. (1 + *x*)9, 4th term
3. , 4th term
4. (3 + *x*)7, 5th term
5. (*x* + 1)20, 3rd term.
6. Expand  in ascending powers of *x* as far as the term in *x*4*.*
7. Use the Binomial theorem to expand:
8. (*x* + *y*)4
9. (*a* – *b*)7
10. (2 + *p*2)6
11. (2h – k)5
12. 
13. 
14. Expand the following using the Binomial theorem
15. (1 + 3*x*)4
16. (2*x* + *y*)4
17. (2 – 3*x*)6
18. Write down and simplify the coefficients of the terms indicated in the expansions of the following
19. , term in *t*4
20. , term in *x*3
21. (2*x* – 3)7, term in *x*5
22. Expand in ascending powers of *x* as far as the term in *x*4.
23. Expand and simplify 
24. Use the Binomial theorem to expand (1 + *x*)12 in ascending powers of *x* up to and including the term in *x*3.
25. Write down the coefficients of the terms indicated in the expansions of the following in ascending powers of *x*
26. (1 + *x*)16, 3rd term
27. (2 – *x*)20, 18th term
28. (3 + 2*x*)6, 4th term
29. , 5th term
30. If *x* is so small that *x*3 and higher powers can be neglected, show that = 64 + 96*x* – 720*x*2
31. The coefficient of *x*3 in the expansion of (1 + *x*)2 is four times the coefficient of *x*2. Find the value of *n*.
32. In the Binomial expansion of , the fourth and fifth terms are equal. find the value of *n*
33. The coefficient of *x*5 in the expansion of (1 + 5*x*)8 is equal to the coefficient of *x*4 in the expansion of (*a* + 5*x*)7. Find the value of *a.*
34. If the first three terms of the expansion of (1 + *ax*)*n* in ascending powers of *x* are 1 – 4*x* + 7*x*2, find *n* and *a*.
35. Use the expansion of (*a* + *b*)4 to evaluate (1.03)4 correct to 2 decimal places.
36. If *x* is so small to allow any term in *x*5 or higher powers of *x* to be neglected, show that (1 + *x*)6(1 – 2*x*3)10 ≈ 1 + 6*x* + 15*x*2 – 105*x*4.
37. When (*1* + *ax*)*n* is expanded in ascending powers of *x*, the expansion is 1 + 2*x* +  + …. Find the values of *n* and *a*.
38. When (1 + *ax*)10 is expanded in ascending powers of *x*, the series expansion is *A* + *Bx* + *Cx*2 + 15*x*3 + …. Find the values of *a*, *A*, *B* and *C*.
39. Find the ratio of the term in *x*7 to the term in *x*8 in the expansion of 
40. Expand the following in ascending powers of *x* as far as the terms in *x*3 and state the values of *x* for which the expansions are valid.
41. (1 + *x*)-2
42. 
43. 
44. 
45. Write down and simplify the term independent of *x* in the expansion of  which is the numerically greatest term in this expansion when *x* = .
46. In the binomial expansion of (1 + *x*)*n*+1, *n* being an integer greater than 2, the coefficient of *x*4 is six times the coefficient of *x*2 in the expansion of (1 + *x*)*n*-1. Determine the value of *n*.
47. Find the value of *n* for which the coefficients of *x*, *x*2 and *x*3 in the expansion of (1 + *x*)*n* are in arithmetical progression.
48. Express  as a sum of three partial fractions and obtain an expansion in ascending powers of *x* of this expression as far as the term involving *x*7.
49. If  denotes the coefficient of *xr* in the expansion of (1 + *x*)*n*, prove that 
50. If the coefficients of *xr-*1,*xr*, *xr*+1 in the binomial expansion of (1 + *x*)*n* are in arithmetical progression. Prove that *n*2 – *n*(4*r* + 1) + 4*r*2 – 2 = 0.
51. Show that if *x* is so small that *x*4 and higher powers of *x* can be neglected, then



1. (a) If , express  and  in terms of *u*.

(b) Assuming that  may be expanded in a series of ascending powers of *x*, obtain the expansion as far as the term in *x*3. Simplify the coefficients.

1. Write down the expansion in ascending powers of *x*. up to the term in *x*2 of (i) 

(ii)  and simplify the coefficients.

Hence or otherwise, expand  in ascending powers of *x* up to the term in *x*2. By using, obtain an estimate, to three decimal places for *π.*

1. Find the term independent of *x* in the expansion of
2.  (b) 

**Answers**

1. (a) 84*x*3 (b) -14080*x*3

(c) 945*x*4 (d) 190*x*2.

1. 1 – . …
2. (a) *x*4 + 4*x*3*y* + 6*x*2*y*2 + 4*xy*3 + *y*4

(b) *a*7 – 7*a*6*b* + 21*a*5*b*2 – 35*a*4*b*3 + 35*a*3*b*4 – 21*a*2*b*5 + 7*ab*6 – *b*7

(c) 64 + 192*p*2 + 240*p*4 + 160*p*6 + 60*p*8 + 12*p*10 + *p*12

(d) 32*h*5 – 80*h*4*c* + 80*h*3*k*2 – 40*h*2*k*3 + 10*hk*4 – *k*5

(e) *x*3 + 3*x* +  + 

(f) 

1. (a) 1 + 12*x* + 54*x*2 + 108*x*3 + 81*x*4

(b) 16*x*4 + 32*x*3*y* + 24*x*2*y*2 + 8*xy*3 + *y*4

(c) 64 – 576*x* + 2160*x*2 – 4320*x*3 + 4860*x*4 – 2916*x*5 + 729*x*6

1. (a)  (b) 540 (c) 6048
2. 7 – 6*x* – *x*2 + 7*x*4 + ….
3. 64*x*5 + .
4. 1 + 12*x* + 66*x*2 + 220*x*3
5. (a) 120 (b) -9120 (c) 4320 (d) 5670.

**11.** 14 **12.** 15 **13.** 2 **14.** 18, -½

**15.** 1.13 **17**. 16,  **18.** , 1, 5, 

**19.** 

# 20. (a) 1 – 2*x* + 3*x*2 – 4x3, -1 < *x* < 1

# DIFFERENTIATION I

Suppose we have a smooth function *f(x)* which is represented graphically by the curve *y* = *f*(*x*). Then we can draw a tangent to the curve at point *P*. It is important to be able to calculate the slope of the tangent of the curve. A graphical method can be used but this is rather imprecise, so we use the following analytical method.

We chose a second point Q on the curve which is near P and join the two points with a tangent line *PQ* called secant and calculate the slope of the line.

Then we can allow Q to approach P so that the secant swings around until it just touches the curve and become a tangent. The limit of the slope of the secant is required to find the slope of the tangent.

Q(*x*+δ*x*, *y*+δ*y*)

P(*x*, *y*)

*y* + δ*y*

*y* = *f*(*x*)

The gradient of the secant PQ =





The gradient of the tangent at P(*f* '(*x*))



as 0

**Example**

Find the gradient of the tangent to the curve *y = x*2.

***Solution***

The gradient of the tangent to the curve *y* = *f*(*x*)



*f*(*x*) = *x*2







= 2*x*

 = 2*x*

|  |
| --- |
| **If *y* = *xn*, then** |

For example: If *y* = *x*4

**

**Example I**

Differentiate the following functions:

1. *x*3 + 2*x*2 + 3*x*
2. 4*x*4 – 3*x*2 + 5
3. *ax*2 + *bx* + *c*

***Solution***

**(a)** *y* = *x*3 + 2*x*2 + 3*x*

= 3*x*3 – 1 + 2 × 2*x*2 – 1 + 3 × 1(*x*1 – 1)

= 3*x*2 + 4*x* + 3*x*0

= 3*x*2 + 4*x* + 3

**(b)** *y* = 4*x*4 – 3*x*2 + 5

 = 4*x*4 – 1 – 2 × 3*x*2 – 1 + 0

= 4*x*3 – 6*x*

**(c)** *y* = *ax*2 + *bx* + *c*

 = 2*ax* + *b*

**Example III**

Find the gradient of the curve *y* = *x*(2 – *x*) at *x* = 2

***Solution***

*y* = *x*(2 – *x*)

*y* = 2*x* – *x*2



= 2 – 2 × 2

= -2

**Example IV**

Find the gradient of the curves at the given points:

1. *y* = (4*x* – 5)2 (, 9)
2. *y* = 3*x*3 – 2*x*2 (-2, -24)
3. *y* = (*x* + 2)(*x* – 4) (3, -5)

***Solution***

**(a) *y* = (4*x* – 5)2**

*y* = 16*x*2 – 40*x* + 25

= 32*x* – 40

= 32 ×  − 40

= 16 – 40

= -24

**(b) *y* = 3*x*3 – 2*x*2**

 = 9*x*2 – 4*x*

= 9 × -22 – 4(-2)

= 36 + 8

= 44

**(c) (*x* + 2)(*x* – 4)**

*y* = *x*2 – 2*x* – 8

 = 2*x* – 2

= 2 × 3 – 2

= 4

**Tangents and Normals to curves**

A tangent is a line which touches a curve at only one point. A normal is a line which is perpendicular to the tangent.



**Example I**

Find the equations of the tangents and normal to the curve at the given points:

1. *y* = *x*2 (2, 4)
2. *y* = 3*x*2 + 2 (4, 50)
3. *y* = 3*x*2 – *x* + 1 (0, 1)
4. 3 – 4*x* – 2*x*2 (0, 1)

***Solution***

1. ***y* = *x*2**

 = 2*x*

= 2 × 2

= 4

The gradient of the tangent = 4

Let *n* be the gradient of the normal

*n* × 4 = -1

*n* = 

Equation of the tangent:



*y* – 4 = 4(*x* – 2)

*y* – 4 = 4*x* – 8

*y* = 4*x* – 4

Equation of the normal:



4(*y* – 4) = 1(*x* – 2)

4*y* – 16 = *x* – 2

4*y* = *x* – 14

1. ***y =* (3*x*2 + 2)**

**** = 6*x*

= 6 × 4

= 24

Gradient of tangent = 24

Let the gradient of the normal be *n*

*n* × 24 = -1

*n* = 

Equation of the tangent:



*y* – 50 = 24(*x* – 4)

*y* – 50 = 24*x* – 96

*y* = 24*x* – 96 + 50

*y* = 24*x* – 46

Equation of the normal:



24(*y* – 50) = -1(*x* – 4)

24*y* – 1200 = -*x* + *4*

24*y* + *x* = 1204

1. ***y* = 3*x*2 – *x* + 1 (0, 1)**

 = 6*x* – 1

 = 6 × 0 – 1 = -1

= -1

Let the gradient of the normal be *n*

*n* × -1 = -1

*n =* 1

Equation of the tangent:

 = -1

*y* – 1 = -*x*

*y* = -*x* + 1

Equation of the normal:

 = 1

*y* – 1 = *x*

*y* = *x* + 1

1. ***y =* 3 *–* 4*x –* 2*x*2(1, -3)**

 = -4 – 4*x*

****= -4 – 4 × 1

= -8

Let the gradient of the normal be *n*

*n* × -8 = -1

*n = *

Equation of the tangent:

 = -8

*y* + 3 = -8(*x* – 1)

*y +* 3 *= -*8*x +* 8

*y = -*8*x* + 5

Equation of the normal:



8(*y* + 3) = *x* – 1

8*y* + 24 = *x* – 1

8*y* + 25 = *x*

**Example II**

Find the coordinates of a point on *y* = *x*2 at which the gradient is 2. Hence find the equation of the tangent to the curve *y* = *x*2 whose gradient is 2.

***Solution***

y = x2

 = 2*x*

2*x* = 2

*x* = 1

If *x* = 1, from *y* = *x*2;

*y* = 12

*y* = 1

The point is (1, 1)

Equation of the tangent:

 = 2

*y* – 1 = 2(*x* – 1)

*y* = 2*x* – 1

**Example III**

Find the equation of the normal to the curve *y* = *x*2 + 3*x* – 2 at the point where it cuts the x*-axis.*

***Solution***

*y* = *x*2 + 3*x* – 2

 = 2*x* + 3

At the *y*-axis, *x* = 0

From *y* = *x*2 + 3*x* – 2,

*y* = 02 + 3 × 0 – 2

*y* = -2

(0, -2)

= 2 × 0 + 3

= 3

The gradient of the tangent = 3

Let the gradient of the normal be *n*

*n* × 3 = -1

*n* = 



3(*y* – -2) = -1(*x*)

3(*y* + 2) = -*x*

3*y* + 6 = -*x*

3*y* + *x* + 6 = 0

**Example IV**

Find the value of *k* for which *y* = 2*x* + *k* is a normal to the curve *y* = 2*x*2 – 3.

***Solution***

*y* = 2*x* + *k*

Comparing y = 2*x* + *k* with *y* = *mx* + *c*;

*m* = 2

∴ Gradient of the normal = 2

*y* = 2*x*2 – 3

 = 4*x*

Let the gradient of the normal be *n*.

4*x* × *n* = -1

*n* = 

Since the gradient of the normal = 2,

 = 2

*x* = 

*y* = 2*x*2 – 3

 − 3

− 3





From *y* = 2*x* + *k*





**Example V**

Find the equations of the tangents to the curve

*y* = (2*x* – 1)(*x* + 1) at the points where the curve cuts the *x*-axis. Find the point of intersection of these tangents.

***Solution***

*y* = (2*x* – 1)(*x* + 1)

*y* = 2*x*2 + *x* – 1

At the *x*-axis, *y* = 0

0 = (2*x* – 1)(*x* + 1)

, *x* = -1

(½, 0) and (-1, 0)

The curve cuts the *x*-axis at (½, 0) and (-1, 0)

*y* = 2*x*2 + *x* – 1

 = 4*x* + 1

= 4 ×  + 1

= 3

= 3

*y* = 3*x* –  …………………….. (i)

= 4 × -1 + 1

= -3

 = -3

*y* = -3(*x* + 1)

*y* = -3*x* – 3 ……………………..(ii)

Equating Eqn (i) and Eqn (ii);





Substituting *x* =  in Eqn (i);

*y* = 3 × 

*y* = 

The two tangents intersect at 

**Example VI**

Find the coordinates of the point on *y* = *x*2 – 5 at which the gradient is 3. Hence find the value of *c* for which the line

*y* = 3*x* + *c* is a tangent to *y* = *x*2 – 5

***Solution***

*y* = *x*2 – 5

 = 2*x*

2*x* = 3 *x* = 

When *x* = ,





*y* = 3*x* + *c*

 satisfies *y* = 3*x* + *c*



**Example VII**

A tangent to the parabola *x*2 = 16*y* is perpendicular to the line *x* – 2*y* – 3 = 0. Find the equation of this tangent and the coordinates of its point of contact.

***Solution***

*x*2 = 16*y*

2*x* *dx* = 16 *dy*



*x* – 2*y* – 3 = 0

*x* – 3 = 2*y*

*y* = 

Since the tangent is perpendicular to the line,

Let the gradient of the tangent be *t.*



*t* = -2

** = -2

*x* = -16

When *x* = -16,

-162 = 16*y*

*y* = 16

(-16, 16)

 = -2

*y* – 16 = -2(*x* + 16)

*y* – 16 = -2*x* – 32

*y* + 2*x* + 16 = 0

The equation of the tangent is *y* + 2*x* + 16 = 0 and the point of contact is (-16, 16)

**Example VIII**

Find the equation of the tangents to the curve *y* = *x*3 – 6*x*2 + 12*x* + 2 which are parallel to the line *y* = 3*x*.

***Solution***

*y* = *x*3 – 6*x*2 + 12*x* + 2

Comparing *y* = 3*x* with *y* = m*x* + c gives *m* = 3

 = 3*x*2 – 12*x* + 12

3*x*2 – 12*x* + 12 = 3

3*x*2 – 12*x* + 9 = 0

*x*2 – 4*x* + 3 = 0

(*x* – 1)(*x* – 3) = 0

*x* = 1 and *x* = 3

If *x* = 1,

*y* = 13 – 6 × 12 + 12 × 1 + 2

*y* = 1 – 6 + 12 + 2

*y* = 9

If *x* = 3, *y* = 33 – 6 × 32 + 36 + 2

*y* = 27 – 54 + 38

*y* = 11

The points are (1, 9) and (3, 11)

 = 3

*y* – 9 = 3(*x* – 1)

*y* – 9 = 3*x* – 3

*y* = 3*x* + 6

 = 3

y – 11 = 3(x – 3)

y – 11 = 3x – 9

y = 3x + 2

**Maximum, Minimum and Inflexion points of a curve**

A Maximum

*I*

*F*

*B* (minimum)

*D*

*E*

*C*

*B* (point of inflexion)

*x* *=a*

*x* *=b*

*x* *=c*

Points A, B, and I are stationary (turning points) of the curve. We say that *f*(*x*) has a maximum value at *x* = *a*, if *f*(*a*) is greater than any value immediately preceding or following, we say that a function *f*(*x*) has a minimum value at *x* = *b*, if *f*(*b*) is less than any value immediately preceding or following.

The tangent to the curve at points A, B and C are horizontal (parallel to the *x*-axis).

The gradient of each tangent to the curve is zero;

*f*(*x*) = 0

At points immediately to the left of the maximum point, *C* the slope of the tangent is positive. i.e. *f* '(*x*) > 0 while points immediately to the right at point D, the slope is negative i.e. *f* '(*x*) < 0.

In other words, at the maximum *f* '(*x*) changes sign from + to (−).

At the minimum point, *f* '(*x*) changes sign from – to +. We can see this at E and F.

Recall *f* '(*x*) = .

|  |  |  |  |
| --- | --- | --- | --- |
|  | Maximum | Minimum | Inflexion |
| Sign of  changes when moving through stationary values. | + 0 − | − + 0 | + 0 +, − 0 − |

To locate maximum, minimum, and inflexion points of a curve without necessarily drawing the curve, we proceed as follows:

1. Find the gradient  of the curve
2. Equate to zero the expression for .
3. Find the values of *x* which satisfy this equation.
4. Consider the sign of  on either sides of these points.
5. Find the value(s) of *y* which correspond(s) to the values of *x*.

**Distinguishing stationary points using the second derivative method**

In order to distinguish the turning points, we find the second derivative.

If  < 0 at (*x*1, *y*2), (*x*1, *y*1) is a point of maximum

If  > 0 at (*x*1, *y*1), (*x*1, *y*1) is a minimum point;

If  = 0 at (*x*1, *y*1), (*x*1, *y*1) is a point of inflexion.

**Example I**

Find the coordinates of the stationary points of the

curve *y* = 2*x*3 – 24*x* and distinguish between them.

***Solution***

*y* = 2*x*3 – 24*x*

 = 6*x*2 – 24

At stationary points,  = 0

****6*x*2 – 24 = 0

*x*2 – 4 = 0

(*x* + 2)(*x* – 2) = 0

*x* = -2 and *x* = 2

If *x* = -2, *y* = 2(-2)3 – 24(-2)

*y* = -16 + 48

*y* = 32

(-2, 32)is a stationary point.

If *x* = 2, *y* = 2(2)3 – 24(2)

*y* = 16 – 48

*y* = -32

(2, -32) is a stationary point

** = 6*x*2 – 24

 = 12*x*

= 12 × -2

= -24 < 0

Since  < 0,  (-2, 32) is a point of maxima.

 = 12*x*

 = 12 × 2

= 24 > 0

Since >0,  (2, -32) is a point of minima.

**Example II**

Investigate the nature of stationary points of the following curves.

1. *y* = *x*(*x*2 – 12)
2. *y* = *x*2(3 – *x*)
3. *y* = *x*(*x* – 8)(*x* – 15)
4. *y* = *x*3(2 – *x*)
5. *y* = 3*x*4 + 16*x*3 + 24*x* + 3

***Solution***

**(a) *y* = *x*(*x*2 – 12)**

*y* = *x*3 – 12*x*

 = 3*x*2 – 12

At a stationary point,  = 0

3*x*2 – 12 = 0

*x*2 – 4 = 0

*x* = ±2

If *x* = 2, *y* = *x*(*x*2 – 12)

*y* = 2(4 – 12)

*y* = 2(-8)

****(2, -16) is a stationary point.

If *x* = -2, *y* = -2(-22 – 12)

*y* = -2(4 – 12)

*y* = -2(-8)

*y* = 16

(-2, 16) is a turning point.

 = 6*x*

 = 6 × 2 = 12

 (2, -16) is a point of minima

= 6 × -2 = -12 < 0

(-2, 16) is a point of maxima.

**(b) *y* = *x*2(3 – *x*)**

*y* = 3*x*2 – *x*3

** = 6*x* – 3*x*2

At a turning point, ** = 0

6*x* – 3*x*2 = 0

3*x*(2 – *x*) = 0

*x* = 0 and *x* = 2

If *x* = 0, *y* = *x*2(3 – *x*)

*y* = 0

(0, 0) is a stationary point.

If *x* = 2, *y* = 22(3 – 2)

*y* = 4

 (2, 4) is a stationary point

Turning points:

 = 6*x* – 3*x*2

= 6 – 6*x*

 = 6

 (0, 0) is a point of minima

 = 6 – 6 × 2

= -6 < 0

 (2, 4) is a point of maxima.

**(c) *y* = *x*(*x* – 8)(*x* – 15)**

*y* = *x*3 – 23*x*2 + 120*x*

 = 3*x*2 – 46*x* + 120

At stationary points,  = 0

3*x*2 – 46*x* + 120 = 0

*x* = 12, *x* = 

If *x* = 12, *y* = *x*(*x* – 8)(*x* – 15)

*y* = 12(12 – 8)(12 – 15)

*y* = 12(4)(-3)

*y* = -144

(12, -144) is a stationary point

When *x* = , 



is a stationary point.

 = 6*x* – 46

= 6 × 12 – 46

= 26 > 0

(12, -144) is a point of minima.



= -26 < 0

 is a point of maxima.

**(d) *y* = *x*3(2 – *x*)**

*y* = 2*x*3 – *x*4

 = 6*x*2 – 4*x*3

At stationary points,  = 0

6*x*2 – 4*x*3 = 0

2*x*2(3 – 2*x*) = 0

*x* = 0, *x* = 

If *x* = 0, *y* = *x*3(2 – *x*)

*y* = 03(2 – 0)

*y* = 0

(0, 0) is a stationary point.

If *x* = *, y* = 

*y* =  = 

 is a stationary point

 = 12*x* – 12*x*2

 = 0

 (0, 0) is a point of inflexion.

= -9

 is a point of maxima

**(e) *y* = 3*x*4 + 16*x*3 + 24*x*2 + 3**

 = 12*x*3 + 48*x*2 + 48*x*

At stationary points,  = 0

12*x*3 + 48*x*2 + 48*x* = 0

12*x*(*x* + 4*x* + 4) = 0

*x* = 0, *x* = -2

If *x* = 0, *y* = 3

(0, 3) is a stationary point.

If *x* = -2, *y* = 3(-2)4 + 16(-2)3 + 24(-2)2 + 3

*y* = 48 – 128 + 96 + 3

*y* = 19

 (-2, 19) is a stationary point.

 = 36*x*2 + 96*x* + 48

 = 48 > 0

(0, 3) is a point of minima.

 = 36(-2)2 + 96(-2) + 48 = 0

(-2, 19) is a point of inflexion.

**Example II**

If *p* = 4*s*2 – 10*s* + 7, find the minimum value of *p* and the values of *s* which gives the minimum value of *p*.

***Solution***

*p* = 4*s*2 – 10*s* + 7

 = *8s* – 10

For minimum value of *p*,  = 0

8*s* – 10 = 0

*s* = 

*p =* 4*s*2 – 10*s* + 7

*p*min = 

*p*min = 

*p*min = 

 = 8*s* – 10

= 8 > 0

*p* is minimum when *S =*   and the minimum value of *p* is .

**Example IV**

A cylindrical can is made so that the sum of the height and the circumference of its base is 45*π* cm. Find the radius of the base of the cylinder if the volume of the can is maximum.

***Solution***

Let the radius of the base be *r* and the height *h* cm.

*h*

(Height + circumference) = 45*π*.

*h* + 2π*r* = 45*π*

*h* = 45*π* − 2*πr* ………………………………. (i)

*V* = *πr*2*h* …………………………….. (ii)

Substituting Eqn (i) in Eqn (ii);

*V* = *πr*2(45*π* - 2*πr*)

*V* = 45*π*2*r*2 - 2*π*2*r*3

 = 90*π*2*r* - 6*π*2*r*2

For the maximum volume,  = 0.

90*π*2*r* - 6*π*2*r*2 = 0

6*π*2*r*(15 – *r*) = 0

*r* = 0 or *r* = 15

But *r* ≠ 0

*r* = 15 cm

**Example V**

Onyango wishes to fence a rectangular farm. He wants the sum of the length and the width of the farm to be 42 cm. Calculate the length and width of the farm for the area of the farm to be as maximum as possible.

***Solution***

*l*

*w*

Let the length and width of the rectangular farm be *l* and *w* respectively.

*l* × *w* = 42

*l* = 42 – *w*

*A* = *l* × *w*

*A* = (42 – *w*)*w*

*A* = 42*w* – *w*2

** = 42 – 2*w*

For the maximum area,  = 0

42 – 2*w* = 0

*w* = 21

*l* = 42 – *w*

= 42 – 21

= 21

**Example VI**

The length of a rectangular block is twice its width, and the total surface area is 108 cm2. Show that if the width of the block is *x* cm, the volume is *x*(27 – *x*2). Find the dimensions of the block if the volume is maximum.

***Solution***

Let the width be *x* cm

2*x*

*x*

*h*

*V = l × w × h*

*V =* 2*x* × *x* × *h*

*V =* 2*x*2*h* ………………………………(i)

Total surface area *A* = 2(*lw + wh + hl*)

108 = 2(2*x*2 + *xh* + 2*xh*)

54 = 2*x*2 + 3*xh*

 = *h* ………………… (ii)

Substituting Eqn (ii) in Eqn (i);





*V* = (27 – *x*2)

For the maximum volume,  = 0

*V* = (27 – *x*2)

*V* = (27*x* – *x*3)

 = (27 – 3*x*2)

For *V*max,  = 0

(27 – 3*x*2)= 0

27 – 3*x*2 = 0

*x*2 = 9

 *x* = 3

*l* = 2*x*

*l =* 6

*h* = 

= 

= 

*h* = 4

**Example VII**

A cylindrical volume *V* is to be cut from a solid sphere of radius *R.* Prove that the maximum volume of the cylindar, *V* is 

***Solution***

Let the height of the cylinder be *h*

*R*

*R*

*h*

*r*

*r*2 +  = *R*2

*r*2 +  = *R*2

*r*2 = *R*2 − 

*V* = *πr*2*h*

*V* = 

*V* = *πR*2*h* – 



For the maximum volume,  = 0

 = 0

*h*2 = 

*h* = 

*V* = *πr*2*h*

*V* = *πr*2

But 

*r*2 = *R*2 – 

*r*2 = R2 – *R*2

*r*2 = *R*2

*V* = *πr*2*h*

*h* = , *r*2 = *R*2

*V*max = 

*V*max = 

**Example VIII**

A cylinder is inscribed in a hemisphere of radius *r* as shown in the figure below.

Find the maximum volume of the cylinder in terms of *r*.

***Solution***

*h*

*r*

*x*2 + *h*2 = *r*2

*x*2 = *r*2 – *h*2

Volume of the cylinder, *V* = *πx*2*h*

*V* = *π*(*r*2 – *h*2)*h*

*V* = *πr*2*h* – *πh*3

= *πr*2 − 3*πh*2

For maximum volume,  = 0

*πr*2 – 3*πh*2 = 0

*π*(*r*2 – 3*h*2) = 0

 = *h*2

*h* = 

*x*2 = *r*2 – *h*2

*x*2 = *r*2 – 

*x*2 = 

*V* = *πx*2*h*

*x*2 =  , *h* = 

*V*max = 

**Example IX**

A rectangular block has a base *x* cm square. Its surface area is 150 cm2. Prove that the volume of the block is (75*x* – *x*3).

**(a)** Calculate the dimensions of the block when the volume is maximum.

**(b)** The maximum volume.

***Solution***

*h*

*x*

*x*

S.A = 2(*lw + wh + hl*)

150 = 2(*x*2 + *xh* + *xh*)

75 = (*x*2 + 2*xh*)

 = *h*

*V* = *l × w × h*

*V = x*2*h*

*V =*

*V =* (75 – *x*2)

*V =* (75*x* – *x*3)

** = (75 – 3*x*2)

For maximum volume,  = 0.

(75 – 3*x*2) = 0

75 – 3*x*2 = 0

*x*2 = 25

*x* = 5

*h* = 

*h = *

*h* = 5

**Example X**

1. A variable rectangular flower garden has a constant perimeter of 40. Find the length of the side when the area is maximum.
2. A variable rectangle has a constant area of 36 cm2. Find the length of the sides when the perimeter is maximum.

***Solution***

*l*

*w*

40

*w*

*l*

Perimeter of the flower garden *P* = 2(*l* + *w*)

40 = 2(*l* + *w*)

20 = *l* + *w*

*l* = 20 – *w*

*A* = *lw*

*A* = (20 – *w*)*w*

*A* = 20*w* – *w*2

 = 20 – 2*w*

For the maximum area,  = 0

20 – 2*w* = 0

*w* = 10

*l* = 20 – *w*

*l* = 10

**(b) *P* = 2(*l* + *w*)**

*lw* = 36

*l =* 

*P* = 

*P* =  + 2*w*

*P =* 72*w*-1 + 2*w*

 = -72*w*-2 + 2

= 

For the maximum perimeter,  = 0

 = 0



*w*2 = 36

*w* = 6

*l* = 6

**Example XI**

Mukasa wishes to enclose a rectangular piece of land of area 1250 cm2 whose one side is bound by a straight bank of a river. Find the least possible length of barbed wire required.

***Solution***

*y*

*y*

*x*

*xy* = 1250

*y* = 

*P = x+ y + y*

*P = x +* 2*y*

*P* = *x* + 

*P = x* + 

**

For the least possible length,  = 0

 = 0

1 = 

*x* = 50

*y* =  = 25

**Example XII**

A closed right circular cylinder of base radius *r* cm and height *h* cm has volume of 54π cm3. Show that *S*, the total surface area of the cylinder, is given by  hence find the radius and height which makes the surface area minimum.

***Solution***

*h*

*r*

*V* = *πr*2*h*

54*𝜋 = 𝜋r*2*h*

 = *h*

Surface area of a cylinder *A* = 2*πr*2 + 2*πrh*

*A* = 2*πr*2 + 2*πr*

*A* = 2*πr*2 + 



For the minimum surface area,  = 0

4πr –  = 0

4*πr*3 − 108*π* = 0

*r*3 = 

*r*3 = 27

*r* = 3



*h* = 6

**Example XIII**

A company that manufactures dog food wishes to pack the feed in closed cylindrical tins. What should be the dimensions of each tin if each is to have a volume of 250*π* cm3 and the minimum possible surface area?

***Solution***

*A* = 2*πr*2 + 2*πrh*

*V = πr*2*h*

250*π* = *πr*2*h*

*h* = 

*A* = 2*πr*2 + 2*πr*

*A* = 2*πr*2 + 

**

For minimum surface area,  = 0

 = 0

*π*(4*r*3 – 500) = 0

*r*3 = 125

*r* = 5 cm

*h* = 

*h =* 10 cm

**Example (UNEB Question)**

Write down the expression of the volume *V* and surface area *S* of a cylinder of radius *r* and height *h*. If the surface area *S* of the cylinder is kept constant, show that the volume of the cylinder will be maximum when *h* = 2*r*

***Solution***

*S* = 2*πr*2 + 2*πrh*

*h* = 

*V* = *πr*2*h*

*V* = 

*V* = (*Sr* – 2*πr*3)

** = (*S* – 6*πr*2)

For maximum volume,  = 0

*S* – 6*πr*2 = 0

*S* = 6*πr*2

*h* = 

*h* = 

*h* = 2*r*

For maximum volume, *h* = 2*r*

**Example (UNEB Question)**

A right circular cone of radius *r* cm has a maximum volume. The sum of its vertical height *h* and circumference of its base is 15 cm. If the radius varies, show that the maximum volume of the cone is cm3.

***Solution***

The base is circular

The circumference of the base = 2*πr*

2*πr* + *h* = 15

*h* = 15 - 2*πr*

Volume of the cone = *πr*2*h*

*V* = *πr*2(15 – 2*πr*)

*= 𝜋*(15*r*2 – 2*πr*3)



For maximum volume,  = 0

(30*r* – 6*πr*2) = 0

30*r* – 6*πr*2 = 0

6*r*(5 – *πr*) = 0

5 = *πr*

*r* = 

*V* = *πr*2*h*

*h* = 15 – 2*πr*; But *r* = 

*h =* 15 – 2π

*h =* 5

*V =* 

*V* = cm3

**Example**

A match box consists of an outer cover open at both ends into which a rectangular box without a top. The length of the box is one and a half times the width. The thickness of the material is negligible and the volume of the match box is 25 cm3. If the width is *x* cm, find in terms of *x* the area of the material used. Hence show that if the least area of the material is to be used to make the box, the length should be 3.7 approximately.

***Solution***

Area of the inner surface = 2(*lw*) + 2(*lh*)

= 

= 3*x*2 + 3*xh*

Area of the water surface = (*lw +* 2*lh +* 2*wh*)

= 

=  + 5*xh*

The total surface of the match box

=  + 5*xh* + 3*xh* + 3*x*2

*A* =  + 8*xh* ………………… (i)

From volume = *l × w × h*,

*V* = 

25 = 

*h* =  ………………………….. (ii)

Substituting Eqn (2) in Eqn (i);

*A* = 

*A* = 

For the least area,  = 0



 = 0

27*x*3 – 400 = 0

*x*3 = 

*x* = 

*l* = 

*l =*  

*l* = 3.68403 cm

*l* ≈ 3.7 cm

## Techniques of Differentiation

**Chain, Product, and Quotient rules**

We can now move to some more properties involved in differentiation. To summarise, so far we have found that:

1. The derivative of a sum is a sum of its derivatives.
2. The derivative of a difference is the difference of the derivatives.

However, it turns out that:

1. The derivative of a product of derivative *f*(*x*)*g*(*x*) is not a product of the derivative.

(*f*(*x*)*g*(*x*)) ≠ *f* '(*x*)*g*'(*x*)

1. The derivative of a quotient is not the quotient of the derivative



1. The derivative of the composition *f*(*x*) is not the composition of the derivatives.

The chain, product and quotient rules tell us how to differentiate in these three situations.

### Chain Rule

The chain rule states that:

|  |
| --- |
|  |

**Example I**

Given that *y* = (*x*2 + 7)100, find 

***Solution***

*y* = (*x*2 + 7)100

Let *t* = *x*2 + 7

 = 2*x*

*y* = *t*100

= 100*t*99



= 100*t*99 × 2*x*

= 200*xt*99

= 200*x*(*x*2 + 7)99.

**Example II**

Given that *y* = (*x*7 – *x*2)42, find .

***Solution***

*y* = (*x*7 – *x*2)42

Let *t* = *x*7 – *x*2  *y* = *t*42



*y* = *t*42

= 42*t*41

*t = x*7 *– x*2

 = 7*x*6 – 2*x*



= 42(*t*41) × (7*x*6 – 2*x*)

= 42(7*x*6 – 2*x*)*t*41

= 42(7*x*6 – 2*x*)(*x*7 – *x*2)41

**Example III**

Find  in terms of *t* in the following expressions:

1. *x* = *t*2, *y* = 4*t* – 1
2. *y* = 3*t*2 + 2*t*, *x* = 1 – 2*t*
3. *x* = 2, *y* = 5*t* – 4
4. *x* = , *y* = *t*2 + 4*t* – 3
5. *x* = , *y* = 

***Solution***

**(a) *x* = *t*2, *y* = 4*t* – 1**

*y* = 4*t* – 1

 = 4

*x* = *t*2

 = 2*t*



 = 4 ×  = 

**(b) *y* = 3*t*2 + 2*t*, *x* = 1 – 2*t***

 = 6t + 2

 = -2



 = (6t + 2) × 

 = -3*t* – 1

**(c)** *x* = , *y* = 5*t* − 4



*y* = 5t – 4

**** = 5



**(d)** *x* = , *y* = *t*2 + 4*t* – 3.

 = -1*t*-1-1 = 

*y* = *t*2 + 4*t* – 3

 = (2*t* + 4)



= (2*t* + 4) × -*t*2

= -2*t*3 – 4*t*2

**(e)** *x* = , *y* = 

**

 = -2(3 + 

= 

*y* = 

** = 





**Example IV**

Find  in terms of *t* if *x* = *at*2 and *y* = 2*at*

***Solution***

*x* = *at*2

 = 2*at*

*y =* 2*at*

 = 2*a*



****  = 2*a* ×  = 

**Example V (UNEB Question)**

A curve is defined by the parametric equations

*x* = *t*2 – *t*

*y* = 3*t* + 4

Find the equation of the tangent to the curve at (2, 10)

***Solution***

*x* = *t*2 – *t* and *y* = 3*t* + 4.



At point (2, 10), *x* = 2 and *y* = 10.

*x* = *t*2 – *t*

*y =* 3*t* + 10

Substituting, for *x* = 2,

2 = *t*2 − *t*

*t*2 − *t* − 2 = 0

*t*2 − 2*t* + t − 2 = 0

*t*(*t* − 2) + 1(*t* − 2) = 0

(*t* − 2)(*t* + 1) = 0

**Either** *t* − 2 = 0,

*t* = 2

**Or** *t* + 1 = 0

*t* = -1

Substituting for *y* = 10,

10 = 3*t* + 4

3*t* = 6

*t* = 2

For 





*y* – 10 = *x* – 2

*y* = *x* – 2 + 10

*y* = *x* + 8

**Example VI**

If *x* = *at*2, *y* = 2*at*, find  and  in terms of *t*.

***Solution***

*x* = *at*2*, y* = 2*at*

 = 2*at* ;  = 2*a*





= 

**Example VII**

A curve is represented parametrically by

*x* = (*t*2 – 1)2; *y* = *t*3

Find 

***Solution***

*x =* (*t*2 – 1)2, *y* = *t*3

 = 2(*t*2 – 1)2*t*

= 4*t*(*t*2 – 1)

*y* = *t*3

 = 3*t*2



= 3*t*2 × 

= 

### Product Rule

Consider *y = uv*, where *v* and *u* are functions of *x*.

*y +*  = (*u* + )(*v* + 

*y +*  = *uv* + *u + v+ *

As 0, 0

0

 + *y* = *uv* + *u*+ *v*

 = *uv + u + v* − *y*

= *uv* + *u + v* – *uv*

 = *u* + *v*

******

As0

***, *** and ******

****

|  |
| --- |
|  |

**Example I**

Differentiate the following

1. (*x*2 + 1)(*x*3 + 2)
2. *x*2(*x* + 1)3
3. 
4. 
5. 
6. 

***Solution***

1. ***y* = (*x*2 + 1)(*x*3 + 2)**

Let *u = x*2 + 1, *v* = *x*3+ 2

****

** =** (*x*2 + 1)(3*x*2) + (*x*3 + 2)2*x*

= 3*x*4 + 3*x*2 + 2*x*4 + 4*x*

= 5*x*4 + 3*x*2 + 4*x*

= 5*x*4 + 3*x*2 + 4*x*

 = 5*x*4 + 3*x*2 + 4*x*.

**(b) *y* = *x*2(*x* + 1)3**

Let *u* = *x*2, *v =* (*x* + 1)3

****

**** = *x*23(*x* + 1)21 + (*x* + 1)32*x*

= *x*(*x* + 1)2[3*x* + 2(*x* + 1)]

= *x*(*x* + 1)2(5*x* + 2)

= *x*(*x*+ 1)2(5*x* + 2)

**(c) *y* = **

*u* = , *v* = 

**





**(d)** *y* = 

Let *u* = *x* – 1, *v* = 







**(e) *y* = **

*y* = 

*y* = 

**

**

**(f) *y* = (1 – *x*)2**

Let *u* = (1 – *x*)2, *v* = (1 – 2*x*)

*y* = *uv*

**







**Example (UNEB Question)**

Given that , find:

1. ****
2. The value of *q* when *R* is maximum.

***Solution***

**(a)** 

Let *u = q*, *v* = ;





**(b)** For *R*max,  = 0

 = 0

1000 – 2*q*2 = 0

*q*2 = 500

*q*2 = 100 × 5

*q* = 

*q* = ±10

*q* = 10 or *q* = -10

### Quotient Rule

Consider *y* = , where *u* and *v* are functions of *x*.

*y* = 





As 0, 0 and , 0



As 0, 





|  |
| --- |
|  |

**Example**

Differentiate the following:

**(a)**  **(b)** 

**(c)**  **(d)** 

**(e)  (f) **

***Solutions***

**(a)** 





*u* = *x*2 + 1; *v*  = *x*2 – 1



**(b)** 

*u* = *x*, *v* = 

**





**(c)** 









**(d) *y* =** 

*y* = 









**(e) **



**









**Example (UNEB Question)**

Differentiate:

**(a) **

**(b) **

***Solution***

**(a) **

*y* = *uv*



**

**(b) *y = ***





**Differentiation of Implicit Functions**

**Example I**

Find  when *x*2 + 2*xy* + *y*2 = 8

***Solution***

(*x*2 + 2*xy* + *y*2) = (8)

2*xdx* + 2(*xdy* + *ydx*) + 2*ydy* = 0

2*x*+2*x* + 2*y* + 2*y* = 0

(2*x* + 2*y*) = -2*x* – 2*y*



**Example II**

If *x*2 – 3*xy* + *y*2 – 2*y* + 4*x* = 0, find 

***Solution***

*x*2 – 3*xy* + *y*2 – 2*y* + 4*x* = 0

(*x*2 – 3*xy* + *y*2 – 2*y* + 4*x*) = (0)

2*x* *dx* – 3(*xdy* + *ydx*) + 2*ydy* – 2*dy* + 4*dx* = 0

2*x* – 3*x* – 3*y* + 2*y* – 2 + 4 = 0

(2*y* – 3*x* – 2) = -4 – 2*x*

 = 

**Example III**

Find  when 3*x*2 – 4*xy* = 7

***Solution***

3*x*2 – 4*xy* = 7

(3*x*2 – 4*xy*) = (7)

6*x* *dx* – 4(*x* *dy* + *y dx*) = 0

6*x* – 4*x* − 4y = 0

6*x* – 4*y* = 4*x*



**Example IV**

If *x*2 + 3*xy* – *y*2 = 0, find  at (1, 1).

Find the equation of the tangent and normal at (1, 1)

***Solution***

*x*2 + 3*xy* – *y*2 = 0

(*x*2 + 3*xy* – *y*2) = (3)

2*x* *dx* + 3(*x* *dy* + *y* *dx*) – 2*y* *dy* = 0

2*x* + 3*x* + 3*y* – 2*y* = 0

(3*x* – 2*y*) = -2*x* – 3*y*



 = -5



*y* – 1 = -5(*x* – 1)

*y* – 1 = -5*x* + 5

*y* = -5*x* + 6 is the equation of the tangent

Let the gradient of the normal be *n*

*n ×* -5 = -1

**

**

5(*y* – 1) = *x* – 1

5*y* – 5 = *x* – 1

5*y* – 4 = *x* is the equation of the normal.

**Example V**

Find the *x*-stationary points of the curve

*x*3 – *y*3 – 4*x*2 + 3*y* = 11*x* + 4

***Solution***

*x*3 – *y*3 – 4*x*2 + 3*y* = 11*x* + 4

(*x*3 – *y*3 – 4*x*2 + 3*y*) = (11*x* + 4)

3*x*2 *dx* – 3*y*2 *dy* – 8*x* *dx* + 3*dy* = 11 *dx*

(3 – 3*y*2) *dy* = (11 – 3*x*2 – 8*x*) *dx*

 = 

At stationar*y* points,  = 0

 = 0

11 – 3*x*2 – 8*x* = 0

3*x*2 + 8*x* – 11 = 0

*x* = 1, *x = *

## Application of Differentiation

### Small Changes

If *A*(*x*, *y*) is a general point in the curve with equation *y* = *f*(*x*) and *B*(*x*+δ*x*, *y*+δ*y*) is a point in the curve close to *A*, then δ*x* is a small increase in *x* and δ*y* is a small increase in *y*

We know from differentiation that



So when  is small, we can say that 



The approximation can be used to estimate the value of a function close to a known value *y* + δ*y* can be estimated if *y* is known.

**Example I**

Given that *y* = 3*x*2 + 2*x* – 4. Use small changes to find the small change in *y* when *x* increases from 2 to 2.02.

***Solution***

*y* = 3*x*2 + 2*x* – 4

 = 6*x* + 2



 = (2.02 – 2) = 0.02

*x* = 2;  = 0.02

****

= (6*x* + 2) 

= [(6 × 2) + 2] × 0.02

= 0.28

**Example II**

Use small changes to estimate 

***Solution***

*y* = 

**



*x* = 100, = 1

**







*x* = 100, **

**

10 + 0.05 = 

10.05 = 

**Example III**

In an experiment, the diameter *x* of a metal is measured and the volume *V* cm3 is calculated using the formula . If the diameter is found to be 10 cm with a possible error of 0.1cm, estimate the possible error in the volume calculated.

***Solution***





= 0.1, *x* = 10



Hence the possible error in the volume is 5*π* cm3

**Example IV**

Find the approximate value of 

***Solution***



*x* = 1000, = 3

**





**

**

**Example I**

Use small changes to find the cube root of 1005

***Solution***



**

*x* = 100, = 5



**

**

**Example**

Use small changes to find .

***Solution***



**

**

**

*x* = 625, = 2





**Percentage Small Changes**

An error of 3% is made in measuring the radius of the sphere. Find the percentage error in the volume.

***Solution***

****

****

****

**Example II**

The height of a cylinder is 10 cm and the radius is 4 cm. Find the approximate percentage increase in the volume when the radius increases from 4 to 4.02 cm.

***Solution***

****

****

*V = 𝜋r*2*h*

*V = 𝜋* (4)2 × 10

*V* = 160*π*

Percentage increase in the volume is 

****

**Example III**

The period *T* of a simple pendulum is calculated from the formula  where *l* is the length of the pendulum and *g* is the acceleration due to gravity constant. find the percentage change in the period caused by lengthening the pendulum by 2%.

***Solution***









Percentage change in period = 



**Example**

An error of 2.5% is made in measuring the area of a circle. What is the percentage error in the circumference?

***Solution***









*C* = 2*πr*



Percentage error in circumference = 



**Example**

If *l* is the length of a pendulum and *t* is the time of a complete swing, it is known that *l = kt*2. The length of the pendulum is increased by *x*%. *x* is so small. Find the corresponding increase in the time of the string.

***Solution***







Percentage increase in time = 



### Rates of Change

**Application of derivatives**

**Example I**

A side of a cube is increasing at a rate of 6cm/s. Find the rate of increase in the volume of the cube when the length of the side is 8cm.

***Solution***

*x*

*x*

*x*

*V* = *x*3

**

**= 6 cm/s



18*x*2



cm3/s

**Example II**

The volume of a cube is increasing at a rate of 2 cm3/s. Find the rate of change of the side of the base when the length is 3 cm.

***Solution***

*l*

*l*

*l*

*V* = *l*3

**= 2cm3/s

**= 3*l*2



2 = 



cm/s

**Example III**

The area of the circleis increasing at a rate of 3cm2/s. Find the rate of change of the circumference when its radius is 2cm.

***Solution***











1.5cm/s

**Example III (UNEB Question)**

A spherical balloon is inflated such that the rate at which its radius is increasing is 0.5cm/s. Find the rate at which:

1. the volume is increasing at the instant when *r* = 5.0cm
2. the surface area is increasing when *r* = 8.5 cm

***Solution***



= 2*πr*2

50*π* cm2/s

*A* = 4*𝜋r*2

**

* =* 8*πr* × 0.5





cm2/s

**Example IV**

A hollow circular cone is held vertex downwards beneath a tap leaking at a rate of 2cm3/s. Find the rise of water level when the level is 6 cm. Given that the height of the cone is 18 cm and its radius is 12 cm.

***Solution***

18cm

12cm



2 cm3/s















**Example V**

An inverted right circular cone of vertical angle 120° is collecting water from a tap at a steady rate of 18π cm3/min. Find:

1. the depth of the water after 12 minutes
2. the rate of increase of the depth at this instant.

***Solution***

*h*

60°

60°

*r*

Volume of the cone *V* = 

cm3/min

1 min 18*π* cm3

12 min *x* cm3

*x* = 12 × 18*π*

= 216*𝜋* cm3





*V* = *πh*3

216 *π* = *πh*3

216 = *h*3

*h* = 6 cm

*V* = *πh*3





cm/min

**Example VI**

An inverted cone with vertical angle of 60° is collecting water leaking from a tap at a rate of 2cm3/s. If the height of water collected is 10cm, find the rate at which the depth is decreasing at that instant.

***Solution***

*h*

*r*

30°

30°









When *h* = 10,



cm/s

**Example**

A hemispherical bowl is being filled with water at a uniform rate when the height of water is *h* cm. The volume is cm3, *r* being the radius of the sphere. Find the rate at which the water level is rising when it is half-way to the top, given that *r* = 6 and the bowl fills in 1 minute.

*h*

*r*

*V =*

When it is full, *r* = *h*

*V =*



144*π* cm

(*Because the bowl fills in a minute*)

When the bowl is not full, 

*r* = 6 cm

**

When *h* = 3, 









**Example**

A horse trough has a triangular cross-section area of height 50 cm and base 60cm and height 2m long. A horse is drinking steadily and when the water level is 5cm below the top, it is being lowered at a rate of 1cm/min. Find the rate of consumption in litres per minute.

***Solution***

*h/*2

*h/*2

*h*

*h* = 50

**

*l* = 200 cm

**

*V* = 100*bh*













4.8 litres/minute

**Example (UNEB Question)**

A hemispherical bowl of radius *a* cm is initially full of water. The water runs out through a small hole at the bottom of the bowl at a constant rate such that it empties the bucket in 24 s. Given that when the depth of water is *x* cm and the volume of water is cm3, show that the depth of water at that instant is decreasing at a rate of *a*3(36(2*a* – *x*))-1. Find how long it will take for the depth of water to be cm and the rate at which the depth is increasing at that instant.

***Solution***

******

*x*

*a*

When it is full of water, *x* = *a*



Because it empties in 24s

24s cm3

1 s *x*

cm3/s

cm3/s

When 









Volume of water in the bowl = 

Volume of the water emptied

= 



1 s

*x* s 

**

*x* = 20.4445 cm

**(b)** 1 +  −  + , -1 < *x* < 1

**(c)**  1 +  +  – , -1 < x < 1

(d) 1 –  +  – , -2 < x < 2

**21**. , 6th **22**. 8 **23**. 7

**24.** , 2*x*3 + 4*x*4 + 4*x*5 + 4*x*6 + 6*x*2

**23 (a)** *u*3 + 3*u*, *u*5 + 5*u*3 + 5*u*

**(b)** 1 + *kx* + 

**29(i)** 1 + *x* − *x*2

**(ii)** **,** 1 + *x* + x2; 3.315

**30(a)**  **(b) -**307.

# CIRCLES

A circle is a 2-dimensional shape in Euclidean geometry made by drawing a curve that is always the same distance from the center

A circle can also be defined as a locus of all points P(*x*, *y*) which are equidistant from the same given point fixed point C(*a*, *b*) [center]

Suppose that the distance of the points P from the given point C (*a*, *b*) is *r*

(*a*, *b*)

*r*

P(*x*, *y*)

*y-b*

*x-a*

*b*

*y*

*a*

*x*

*x-*axis

*y-*axis

*C*

is the equation of the circle wit center (a, b) and radius r

If the center C is (0, 0) then the equation of the circle is

For

Suppose ,



The equation of the circle becomes

is the standard equation of a circle with center (-*g*, -*f*) and radius

***Example I***

Find the center and the radius of the circles below

***Solution***

Comparing with

,

 The center is C (1, 2) and

is a circle with radius 3 units and center (1, 2)

Compare  with 

*r* = 5

The center is (-1, 3)

∴ is the equation of the circle with center (-1, 3) and radius 5.

*x*2 + *y*2 – 4*x* – 2*y* – 4 = 0

Comparing with

Since the center is (-g, -f),

The center is (2, 1)

Radius =

is a circle with radius 3 units and center (2, 1)

Comparing with

2*gx* = -*x*, 2*fy* = *y*, 

, 

Center (-*g*, -*f*)

Centre (

Radius

Radius

Radius

=

= 1

is the equation of the circle with center and radius 1.

**Example III**

Find the equation of the circle with the following centers and radii

1. Center (2, 3) radius 1
2. Center (3, -4) radius 5
3. Center and radius
4. Center and radius
5. Center (0, -5) and radius 5

***Solution***

1. Center (2, 3) radius 1

Given a circle of centre (*a*, *b*) and radius *r*. The equation of the circle is (*x* – *a*)2 + (*y* – *b*)2 = *r*2.

Consider the equation of the circle

with center (*a*, *b*) and radius *r*

The equation of the circle with center (2, 3) and radius 1 is

1. Center (3, -4) radius 5

The equation of the circle with center (3, -4) and radius 5 is

1. Center and radius

Equation of the circle with center and radius is 

1. Center and radius
2. Center (0, -5) and radius 5

**Example III**

State which of the following are equations of the circles

***Solution***

is an equation of a circle

is not an equation of a circle, since is not real.

is an equation of the circle when *c* < 0.

Comparing with

 is not an equation of a circle because of the component of

is a circle

Is not a circle because the co-efficient of are not the same

Is not a circle because the co-efficient of *x*2 and *y*2 are not the same.



Is a circle

Is not a circle

**Example IV (UNEB Question)**

The equation of the circle with center O is given by where *A*, *B* and *C* are constants. Given that 4*A* = 3*B*, 3*A* = 2*C* and *C* = 9

Determine

1. The coordinates of the center of the circle
2. The radius of the circle

***Solution***

4A = 3B …………………… (1)

3A = 2C …………………… (2)

C = 9 ………………………. (3)

Substituting eqn. (3) in eqn. (2)

Substituting *A* = 6 in Eqn (1)

4 × 6 = *3B*

Comparing with

*2g* = 6  *g* = 3

*2f* = 8, *f* = 4

*C =* 9

Centre (-3, -4)

Radius =

Radius =

=

= 4

**Example V**

Find the equation of a circle whose center is (2, 1) and passes through (4, -3)

***Solution***

(2, 1)

(4, -3)

**Example VI**

The points (8, 4) and (2, 2) are end points of the diameter of the circle. Find the center, the radius and the equation of the circle

***Solution***

(8, 4)

(2, 2)

(5, 3)

(8, 4)

**Example VI**

Find the equation of a circle passing through points (2, 3) and (4, 5) having its center on the line

***Solution***

Let the center be (x, y). Since it lies on the line

, let the *x*-co-ordinate of the center be ***a***.

Then the *y*-co-ordinate

*y* = 4*x* + 3

(2, 3)

(*a*, 4*a*+3)

(4, 5)





*r*1 = *r*2 = *r*



17*a*2 – 4*a* + 4 = 17*a*2 – 24*a* + 20

20*a* = 16





Centre(*a*, 4*a*+3)

centre 

centre 





5*x*2 + 5*y*2 – 8*x* – 62*y* + 137 = 0

**Example**

What is the equation of the circle whose center lies on the which touches the positive axes.

***Solution***

Let the *y*-coordinate of the centre be *a*

*x* – 2y + 2 = 0

*x* – 2*a* + 2 = 0

*x* = 2*a*– 2

(2*a* – 2, *a*)

(0, 2*a* - 2)

*x* – 2*y* + 2 = 0

(2*a* - 2, *a*)

*a*

*r*

2*a* – 2 = *a*

*a* = 2

The center is (2, 2); radius *r* = 2

**Equation of circle passing through three points**

**Example I**

Find the equation of the circle passing through the points

1. A(-2, 1) B(6, 1) and C(-2, 7)
2. A(-1, 4) B(2, 5) and C(0, 1)
3. A(3, 1) B(8, 2) and C(2, 6)
4. A(5, 7) B(1, 6) and C(2, 2)

***Solution***

1. A(-2, 1) B(6, 1) and C(-2, 7)

(*a*, *b*)

(-2, 7)

(-2, 1)

(6, 1)



Equating the radii; *r*1 = *r*2 = *r*



*a*2 + *b*2 + 4*a* – 2*b* + 5 = *a*2 + *b*2 – 12*a* – 2*b* + 37

Also *r*1 = *r*3 = *r*

*b*2 – 2*b* + 1 = *b*2 – 14*b* + 49

Center (*a*, *b*) = (2, 4)

(*x* – *a*)2 + (*y* – *b*) = *r*2

***Alternatively***; Consider the general equation of the circle

At (-2, 1)

…………………. (1)

At (6, 1), 

……………….. (2)

At (-2, 7), 

……………… (3)

Solving equation (1), 2 and 3 simultaneously

Substituting in the general equation of the circle

(As before)

1. **A(-1, 4) B(2, 5) and C(0, 1)**

(-1, 4)

(*a*, *b*)

(0, 1)

(2, 5)

*r*

*r*1 = *r*2= *r*



………………………… (1)

Similarly, *r*1 = *r*3 = *r*

…………………………. (2)

From eqn. (1)

Substituting in eqn. (2)

*b* = 6 – 3 × 1

Center (1, 3)

(*x* – *a*)2 + (*y* – *b*)2 = *r*2

***Alternatively***

At (-1, 4): 

At (2, 5): 

………………… (2)

At (0, 1):

…………………………. (3)

Solving eqn. 1, 2 and 3 simultaneously

Substituting in the general equation of the circle

(as before)

1. A(3, 1) B(8, 2) and C(2, 6)

(3, 1)

(*a*, *b*)

(8, 2)

(2, 6)

*r*

*r*1 = *r*2 = *r*



………………………….. (1)

Similarly; *r*1 = *r*3 = *r*



…………………………. (2)

Substituting equation (1) in (2)

Center (5, 4)

***Alternatively***

At (3, 1): 

…………………… (1)

At (8, 2): 

…………………. (2)

At (2, 2): 

……………………. (3)

Solving eqn. 1, 2 and 3 simultaneously

Substituting the values of g, f and c in the general equation

(As before)

1. A(5, 7) B(1, 6) and C(2, 2)

*r*

(2, 2)

(5, 7)

(1, 6)

(*a*, *b*)

Equating the radii

*r*1 = *r*2 = *r*

……… (1)

……… (2)

From equation (1)

………………………… (3)

From eqn. (2)

(*a* – 2)2 + (*b* – 2)2 = (*a* – 5)2 + (*b* – 7)2

………………………. (4)

Solving Eqn (3) and (4) simultaneously

, and

Center

(*x* – *a*)2 + (*y* – *b*)2 = *r*2

***Alternatively***

At (5, 7): 

………………… (1)

At (1, 6): 

………………… (2)

At (2, 2): 

…………………… (3)

Solving eqn. 1, 2 and 3 simultaneously

Substituting g, f and c in the general equation of the circle

.

## Parametric Equations of circle

Consider a circle the parametric equations of the above circles are and

**Example I**

Find the parametric equation of the circle

***Solution***

The parametric equations of the circle are

**Example II**

Find the parametric equations of the circle

***Solution***

Comparing with the equation of the circle

**Example III**

Find the parametric equations of the circle

***Solution***

By completing squares;s

**Example IV**

Find the parametric equation of a circle

Solution

By completing squares;



**Example V**

Find the Cartesian equation of the circle with parametric equations

***Solution***



1

But 

**Tangents to the Circle**

A tangent to the circle is a line which touches the circle at only one point and makes with the radius of the circle.

**Length of the tangent to a circle**

**Example**

Find the length of the tangent from (5, 7) to the circle

***Solution***

Comparing with

.

Center (-*g*, -*f*)

Center (2, 3)

2

A

C(2, 3)

B(5, 7)

The length of the tangent is  units

**Example II**

Find the lengths of the tangents from the given points to the following circles



***Solution***

Comparing with

.

Center (-*g*, -*f*)

Center (2, 3)

C(2, 3)

A(0,0)

B



Comparing with .

Center (-3, -5)

C(-3, -5)

A(-2,3)

B

**Alternative method of finding length of the tangent to a circle**

The length of a tangent drawn from a point to the circle is given by

where *L* = length of the tangent

The square of the length of the tangent from the point P is called a power point with respect to the circle.

**Example I**

Find the length of the tangent drawn from the point (5, 1) to the circle

***Solution***

Comparing with



**Example II**

If the length of the tangent from the point (f, g) to the circle is four times the length of the tangent from (*f*1, *g*1) it to the circle show that 

***Solution***

**

For *x*2 + *y*2 – 4*x* = 0, *g* = -2 and *f* = 0





But *L*1 = 4*L*2







(as required)

**Equation of a Tangent**

**Example I**

Find the equation of the tangent to the circle

***Solution***

***Alternatively***

**Note:** The equation of the tangent to the circle is

The equation of the tangent to the circle at is

We can now find the equation of the tangent to the

Comparing with

**Example II**

Find the equation of the tangent to the circle at C (1, -1)

***Solution***



***Alternative method***

From 2*x*2 + 2*y*2 – 8*x* – 5*y* – 1 = 0,

*x*2 + *y*2 – 4*x* –  − = 0

*x*2 + *y*2 + 2*gx* + 2*fy* + *c* = 0

*g* = -2, *f* = , *c* = 

*x*1 = 1, *y*1 = -1

The equation of the tangent is given by

*xx*1 + *yy*1 + *g*(*x* + *x*1) + *f*(*y* + *y*1) + *c* = 0

*x*(1) + *y*(-1) + -2(*x* + 1) + (*y* – 1) +  = 0

*x* – *y* – 2*x* – 2 +  + = 0



-4*x* – 9*y* – 5 = 0

4*x* + 9*y* + 5 = 0

**Example III**

The tangent to the circle *x*2 + *y*2 – 4*x* + 6*y* – 77 = 0 at the point (5, 6) meets the axes at A and B. find A and B

***Solution***



*x* + 3*y* = 23

***Alternative method***

Comparing *x*2 + *y*2 – 4*x* + 6*y* – 77 = 0 with *x*2 + *y*2 + 2*gx* + 2*fy* + *c* = 0

*g* = -2, *f* = 3, *c* = -77

*x*1 = 5, *y*1 = 6

The equation of the tangent is

*x*1*x* + *yy*1 + *g*(*x* + *x*1) + *f*(*y* + *y*1) + *c* = 0

5*x* + 6*y* + -2(*x* + 5) + 3(*y* + 6) – 77 = 0

5*x* + 6*y* – 2*x* – 10 + 3*y* + 18 – 77 = 0

3*x* + 9*y* = 69

*x* + 3*y* = 23, as before.

At the *x*- axis (A), *y* = 0

The tangent meets the x- axis at (23, 0)

At the *y*- axis (B), *x* = 0

The curve cuts the y- axis at (0,

**Example VII**

Find the equation of the tangent to the circle

***Solution***



***Alternatively***

Given a circle the equation of the tangent at is

Comparing with

, *x*1 = 4, *y*1 = -1



**Example IV**

Show that is a tangent to the circle if

***Solution***

**Example V**

Show that the line *y* = *x* + 1 touches the circle

.

***Solution***

*y* = *x* + 1

For the line to touch the circle

The line y = x + 1 touches the circle

**Note:**

|  |
| --- |
| If is a line and is a circle then   1. the line is a secant to the circle 2. If the line touches the circle 3. If the line doesn’t meet the circle |

**Example VI**

For what values of *c* will the line *y* = 2*x* + *c* be tangent to the circle

***Solution***

For tangency

**Example VII**

For what values of does the line touch the circle ?

***Solution***

…………. (i)

*x*2 + *y*2 – 10*x* = 10 ………. (ii)

Substituting in Eqn (ii)

For tangency

**Example VIII**

Find the equation of the tangents to the circle

which are parallel to the line

***Solution***

Let the tangent be

Since the tangent is parallel to

is equation of the tangent

Comparing with

Center (+3, -2)

(3, -2)

B

*r*

4*x* + 3*y* = 3*c*

But we can obtain *r* using the formula for perpendicular distance of a point from a line

Since the tangents to the circle are given by

The equations of the tangents are 4*x* + 3*y* = -19 and

**Example ix**

(i) Find the equation of the tangents to the circle which are parallel to line

(ii) Which are perpendicular to the line 3*x*–4*y* – 1=0

***Solution***

Comparing with

Center (1, 2)

Since the tangents are parallel to the line

for the tangent

are the equations of the tangents

(1, 2)

B

3

-3*x* + 4*y* = 4*c*

Since the equations of the tangent that are parallel to the line are

The required tangents are:

**(ii)** Let the tangents that are perpendicular to the line be







Center (1, 2)

(1, 2)

B

3

4*x* + 3*y* = 3*c*

Since the tangent are;

**Director Circle**

The locus of the point of intersection of two perpendicular tangents is called the Director circle of a given circle. The Director circle of a circle is a concentric circle having radius equal to times the original radius.

**Example**

Find the equation of the director circle of the circle

***Solution***

Center (2, -1)

Radius *r* =

The center of the director circle is (2, -1) and the radius of the director circle is

The equation of the director circle is

**Example II**

Find the equation of a director circle of the circle whose diameters are and and has an area of 154.

***Solution***

…………………… (1)

………………………. (2)

Solving eqn. (1) and (2) simultaneously

The center of a circle is (-3, 2)

Radius of the director circle is

The equation of the director circle is

Therefore, is the equation of the director circle.

**Equation of a common chord of two circles**

Let the equations of two intersecting circles be

…… (1)

And

……. (2)

Intersect at and

P(x1, y1)

Q(x2, y2)

Now we observe from the figure that lies on both given equations therefore, we get

…… (3)

….. (4)

Eqn. (3) − Eqn. (4)

… (5)

Again we observe from the above figure that point lies on both circles

…... (6)

…... (7)

Eqn. 6 – eqn. 7

…… (8)

From eqn. 5 and 8, it’s evident that the points and lie on which is a linear equation in x and y.

**Note:** While finding the equation of the common chord of two given intersecting circle, we fast express each equation in the form

**Example**

Determine the equation of the chord of the two intersecting circles and and prove that the common chord is perpendicular to the line joining the two centres of the circles.

***Solution***

……….. (1)

………… (2)

Eqn. (1) – Eqn. (2)

The equation of the chord:

The gradient of the chord is

Comparing with

Center (2, 1) = *C*1

Comparing with

Center ( = *C*2

The gradient joining the two centers

Gradient of chord × gradient of line joining the two centres

The chord is perpendicular to the line joining the two centers

**Example**

Show that the common chord of the circles and passes through the origin

***Solution***

……………………. (1)

………... (2)

Eqn. (2) – eqn. (1)

is the equation of the common chord

At (0, 0),

The common chord passes through the origin.

**Example**

Find the equation of the common chord of the circles

***Solution***

………… (1)

………. (2)

Eqn. (2) – eqn. (1)

**Example**

Find the point of intersection of the two circles

and

***Solution***

When we are finding the point of intersection, we first find the equation of the common chord and then we solve it simultaneously with one of the equations of the circles

……….. (1)

……… (2)

Eqn. (1) – eqn. (2)

is the equation of the common chord

Substituting in eqn. (1)

The point of intersection of both circles is

**Example**

Find the point of intersection of the circles

and 

***Solution***

…………. (1)

………. (2)

Eqn (1) – Eqn (2)



The point of intersection is (3, 2)

**Types of intersecting circles**

1. Touching each other internally

Two circles touch each other internally if the distance between their centers is equal to the distance between their radii

C2

C1

*r*2

*r*1

1. Circle intersect at two distinct points when

C2

C1

1. Concentric circles

C

These are circles with the same center.

1. Circle which touches each other externally if the distance between their centers is equal to the sum of their radii.

*r*1

*r*2

**Example**

Prove that the circles  and  touch each other externally.

***Solution***

Comparing with

Center

Radius

Comparing with

Center (-1, -1)



Since

The two circles touch each other externally

**Orthogonal Circle**

Two circles are said to be orthogonal if the tangents at their point of intersection cut at right angles as illustrated below.

C1

C2

r1

r2

**Example**

Prove that the circles and are orthogonal

***Solution***

Comparing with

Center

Similarly

Comparing with

Center

(-2,1)

(2,4)

4

3

Since

The two circles are orthogonal

**Example (UNEB Question)**

**13**. a) Form the equation of a circle that passes through the points *A* (-1, 4), *B* (2, 5) and *C* (0, 1)

**b**) The line *x* + *y* = *c* is a tangent to the circle

*x*2 + *y*2 −4*y* + 2 = 0. Find the coordinates of the point of contact of the tangent for each value of *c*.

***Solution***

General equation of the circle is

*x*2 + *y*2 + 2*gx* + 2*fy* + *c* = 0

At *A*(-1, 4),

-2*g* + 8*f* + *c* = -17............................. (i)

At *B*(2, 5);

4*g* +10*f* + *c* = -29............................. (ii)

At *C*(0, 1):

2*f* + *c* = -1........................................ (iii)

2 Eqn (i) + Eqn (ii)

10*c* = 50

*c* = 5

From Eqn (iii);

2*f* + 5 = -1

2*f* = -6

*f* = -6

From Eqn (i)

-2*g* +8(-3) + 5 = -17

-2*g* = 24 − 17 − 5

*g* = - 1

Hence the equation of the circle is

***x*2 + *y*2 − 2*x* − 6*y* + 5 = 0**

***Alternatively***



(0 − *x*)2 + (1 − *y*)2 = (-1 − *x*)2 + (4 − *y*)2

3*y* − *x* = 8................................. (i)

Also

(0 − *x*)2 + (1 − *y*)2 = (2 − *x*)2 + (5 − *y*)2

2*y* + *x* = 7...................................... (ii)

Eqn (i) + Eqn (ii)

5*y* = 15

*y* = 3

3(3) − *x* = 8

*x* = 1

Centre of the circle = (1, 3) and the radius is



Equation of the circle is ***x*2 + *y*2 − 2*x* − 6*y* + 5 = 0**

**b**) *x*2 + *y*2 − 4*y* + 2 = 0,

And *y* = *c* − *x*

At the point of contact,

*x*2 + (*c* − *x*)2 − 4(*c* − *x*) + 2 = 0

2*x*2 + (4 − 2*c*)*x* + (*c*2 − 4*c* + 2) = 0

For tangency, *b*2 = 4*ac*

(4 − 2*c*)2 = 4×2 × (*c*2 − 4*c* + 2)

4(2 − *c*)2 = 8 (*c*2 − 4*c* + 2)

(2 − *c*)2 = 2(*c*2 − 4*c* + 2)

4 − 4*c* + *c*2 = 2*c*2 − 8*c* + 4

*c*2 − 4*c* = *c*(*c* − 4) = 0

Either *c* = 0 or *c* = 4

If *c* = 0, *y* = -*x*

⟹ *x*2 + *x*2 + 4*x* + 2 = 0

2*x*2 + 4*x* + 2 = 0

*x*2 + 2*x* + 1 = (*x* + 1)2 = 0

⟹ *x* = -1

Therefore *y* = 1

The point is (-1, 1)

If *c* = 4, *y* = 4 − *x*

(4 − *x*)2 + *x*2 − 4(4 − *x*) + 2 = 0

16 − 8*x* + *x*2 + *x*2 − 16 + 4*x* + 2 = 0

2*x*2 − 4*x* + 2 = 0

*x*2 − 2*x* + 1 = (*x* − 1)2 = 0

*x* = 1, *y* = 3

The point is (1, 3)

**Example (UNEB Question)**

**a)** Find the equation of a circle which passes through the points (5, 7), (1, 3) and (2, 2).

**b**) i) If *x* = 0 and *y* = 0 are tangents to the circle, *x*2 + *y*2 + 2*gx* + 2*fy* + *c* = 0, show that *c* = *g*2 = *f* 2.

**ii**) Given that the line 3*x* – 4*y* + 6 = 0 is also a tangent to the circle in (b) (i) above, determine the equation of the circle lying in the first quadrant. (06 *marks*)

***Solution***

**(a)** The equation of the circle is given by;

*x*2 + *y*2 + 2*gx* + 2*fy* + *c* = 0

Substituting for (5, 7),

25 + 49 + 10*g* + 14f + *c* = 0

74 + 10*g* + 14f + *c* = 0

10*g* + 14*f* + *c* = -74 …………..(i)

Substituting for (1, 3)

1 + 9 + 2*g* + 4*f* + *c* = 0

2*g* + 6*f* + *c* = -10 ……………(ii)

Substituting for (2, 2)

4 + 4 + 4*g* + 4*f* + *c* = 0

4*g* + 4*f* + *c* = -8........................... (iii)

Eqn (i) − Eqn (ii)

6*g* + 10*f* = -64

*g* + 6*f* = -8................................... (iv)

Eqn (i) − Eqn (iii)

6*g* + 10*f* = -66

3*g* + 5*f* = -33................................. (v)

3 Eqn (iv) − Eqn (v)

3*g* + 3*f* = -24

3*g* + 5*f* = -33

-2*f* = 9



Substituting for *f* in Eqn (iv)



Substituting for *f* and *g* in Eqn (iii)



-28 − 36 + 2*c* = -16

2*c* = 64 − 16



The equation of the circle is *x*2 + *y*2 − 7*x* − 9*y* + 24 = 0

**b**) Given *x*2 + *y*2 + 2*gx* + 2*fy* + *c* = 0

When *y* = 0, *x*2 + 2*gx* + *c* = 0

For tangency, *b*2 = 4*ac*

(2*g*)2 = 4*c*

4*g*2 = 4*c*

*g*2 = *c*

When *x* = 0, *y*2 + (2*f*)2 + c = 0

For tangency, *b*2 = 4*ac*

(2*f*)2 = 4*c*

4*f* 2 = 4*c*

*f* 2 = *c*

Hence *c* = *g*2 = *f* 2

**ii)** From the line 3*x* − 4*y* + 6 = 0

4*y* = 3*x* + 6



Substituting for *y* and *y*2 in the equation

*x*2 + *y*2 + 2*gx* + 2*fy* + *c* = 0



16*x*2 + (3*x* + 6)2 + 32*fx* + 8*f*(3*x* + 6) + 16*f* 2 = 0

16*x*2 + 9*x*2 + 36*x* + 36 + 32*fx* + 24*fx* + 48*f* + 32*f* 2 = 0

25*x*2 + (36 + 54*f*)*x* + (36 + 48*f* + 16*f* 2) = 0

For tangency, *b*2 = 4*ac*

(36 + 54*f*)2 = 4 × 24(36 + 48*f* + 16*f* 2)

(36 + 54*f*)2 = 100(36 + 48*f* + 16*f* 2)

By opening brackets and simplifying we obtain

2*f* 2 − *f* − 3 = 0

2*f*2 − 3*f* + 2*f* − 3 = 0

*f*(2*f* − 3) + 1(2*f* − 3) = 0

(2*f* − 3)(*f* + 1) = 0

Either 2*f* − 3 = 0

2*f* = 3

*f* = 3/2

Or *f* + 1 = 0

*f* = -1

Now *f* = *g*

⟹ *g* = 3/2 or -1

Centre of the circle is (-*g*, -*f*). Since it is in the first quadrant, then the centre is (1, 1)

But *c* = *g*2 = *f* 2 = 1

The equation of the circle is *x*2 + *y*2 − 2*x* − 2*y* +1 = 0

# LOCI

When a point moves in the plane according to some given conditions, the path along which it moves is called a locus.

A locus is a set of points which satisfy certain geometric conditions. Many geometric shapes are most naturally and easily described as a loci. For example a circle is a set of points in the plane which are fixed at distance r from a given point *P* (center).

Problems involving describing a certain locus is often solved by explicitly finding equations for the coordinates of the points in the locus. Here is a step by step procedure for finding plane loci

**Step I:** If possible, choose a coordinate system that will make computations and equations as simple as possible

**Step II:** Write the given conditions in mathematics from involving the coordinates (*x*, *y*).

**Step III:** Simplify the equations.

**Step IV:** Identify the shape out by the equations.

**Example I**

Find the locus of a circle with center at the origin and radius 5 units.

***Solution***

(0,0)

P(*x*,*y*)

5 units

*x*

*y*

The locus is *x*2 + *y*2 = 5

**Example II**

What is the locus of a point which moves so that its distance from the point (3, 1) is 2 units?

***Solution***

P(*x*, *y*)

2 units

(3, y)

*y*

x

The locus is a circle with center (3, 1) and radius 2

**Example III**

What is the locus of point which is equidistant from the origin (0, 0) and the point (-2, 5)

***Solution***

(0, 0)

P(*x*, *y*)

*x*

*y*

A(-2, 5)

The locus is a straight line with a positive gradient.

**Example IV**

Find the locus of a point which is equidistant from the line and the origin.

***Solution***

A

P(*x*, *y*)

(0, 0)

*x* = -1

(*x* + 1 = 0)

x

*y*

The perpendicular distance of the line *ax + by + c=*0 from is

Comparing with

The perpendicular distance of the point from the line is

The locus is a parabola

**Example**

Find the locus of a point which is equidistant from the point (0, 1) and the line

A

*y*

*y* = -1

*x*

P(*x*, *y*)

(0, 1)

Comparing with general equation of the line

The perpendicular distance of the point from the line is

The locus is a parabola

**Example VI (UNEB Question)**

A point p is twice as far from the line as from the point (3, 0). Find the locus of P.

***Solution***

A

*x*

*y*

*P*(*x*, *y*)

B(3, 0)

*x*+*y* = 5

The perpendicular distance of point from the line is





**Example VII**

Find the locus of a point which moves so that the sum of squares of its distances from and

(2, 0) is 26

P(*x*, *y*)

A(-2, 0)

B(2, 0)

***Solution***



The locus is a circle with center and radius 3 units

**Example VIII**

Find the locus of the point P which moves so that its distance from the point ( is a half its distance from the line

***Solution***

P(x, y)

B(5, 0)

y

x

A

*x* = 8

2*PB* = *PA*

**Example IX**

Find the locus of a point which is equidistant from the line *x* = 2 and the circle

***Solution***

A

*x* = 2

*x*

P(*x*, *y*)

y

*x*2 + *y*2 = 1

(0, 0)

B

The locus is a parabola

**Example X**

The points *R*(2, 0)and P lie on the *x*-axis and *Q*(0, -*y*)on the y- axis. The perpendicular from the origin to *QR* meets *PQ* at point S(*X*, -*Y*). Find the locus of S.

***Solution***

Q(0, -*y*)

S(*X*, -*Y*)

P(3, 0)

*R*(2, 0)

*y*

*x*

O

Since *S* is in terms of X and Y

Then the locus of S must be in terms of X and Y

From the figure above,

(The gradient of PQ) = (Gradient of SQ)



…………….. (1)

(Gradient of RQ) X (Gradient of OS) = -1

……………………...(2)

Substituting eqn. 2 in (1)

is the locus of S

**Example XI**

Variable lines through the point O(0, 0) and A(2, 0) intersect at right angles at the point *P*. Show that the locus of the midpoint of OP is

***Solution***

O(0, 0)

P(*x*, *y*)

A(2, 0)

M(*X*, *Y*)

(The gradient of *OP*) × (Gradient *AP*) = -1

………………………… (1)

Let the midpoint Op be M(*X*, *Y*)

But *x* and *y* satisfy the above equation.

Substituting *x* = 2*X* and *y* = 2*Y* in Eqn (1);

**Example XII**

P is a point on a line of length 12 units, which moves so that it’s length lie on the axes. Find the locus of P when its

1. The midpoint of line,
2. The point of trisection near the *y-*axis.

***Solution***

y

x

(0, 0)

L(X, 0)

P(*x*, *y*)

M(0, Y)

ML = 12 Units

Since P is a midpoint of LM

Similarly,

Applying Pythagoras theorem on triangle OLM

……………………. (1)

Substituting in equation (1)

The locus of p is a circle with center (0, 0) and radius 6

(0, 0)

(X, 0)

(0, Y)

P( , )

P( , )

Since P(*x*, *y*) is a point of intersection near the *y*-axis

Substituting in Eqn (1)

**Example XIII**

The fixed points *A* and *B* have coordinates (-3*a*, 0) and (*a*, 0) respectively. Find the locus of P which moves in the coordinate plane so that *AP* = 3*pB*. Show that the locus is a circle, S which touches the axis of *y* and has a center at the point. A point Q moves in such a way that the perpendicular distance of Q from the y-axis is equal to the length of the tangent from Q to the circle S. find the equation of the locus of Q. show that this locus is also a locus of points which are equidistant from the line 4*x* + 3*a* = 0 and the point .

***Solution***

P(*x*, *y*)

B(*a*, 0)

A(-3*a*, 0)



Comparing

Center

S



C( , 0)

Q(*x*, *y*)

A

A (3*a*/4, 0)

X = 

P(*x*, *y*)

B( , *y*)



## Revision Exercise

1. Find the equation of the circle which passes through the origin and the points (2, 0), (3, -1).
2. Fid the radii and coordinates of the centres of the following circles.
3. *x*2 + *y*2 + 4*x* – 6*y* + 12 = 0
4. *x*2 + *y*2 – 2*x* – 4*y* + 1 = 0
5. *x*2 + *y*2 – 3*x* = 0
6. *x*2 + *y*2 + 3*x* – 4*y* – 6 = 0
7. Find the equations of the circle with the following centres and radii:
8. (3, 2), 4
9. (-1, -2), 1
10. (0, 0), 5
11. (½, 0), 
12. (4, -1), 
13. Find the equation of the circle which has the points (0, -1) and (2, 3) as ends of its diameter.
14. What is the equation of the circle with centre (2, -3) and touches the *x-*axis?
15. Find the equation of the curve having *AB* as diameter where A is the point (1, 8) and B(3, 14).
16. Find the range of the values of *k* for which each of the following represents a circle with non-zero radius.
17. *x*2 + *y*2 = *k*
18. *x*2 + *ky*2 – 2*x* – 8 = 0
19. *kx*2 + *y*2 + 4*y* + 9 = 0
20. 2*x*2 + 2*y*2 + *kxy* – 9 = 0
21. Find the equation of the diameter of the circle *x*2 + *y*2 – 6*x* + 2*y* = 15, which when produced passes through the point (8, -2).
22. Find the radii of the two circles with centres at the origin which touch the circle *x*2 + *y*2 – 8*x* – 6*y* + 24 = 0
23. Find the equation of the tangent to the circle (*x* – 2)2 + (*y* – 3)2 = 16 at the general point ((2 + 4cos*θ*), (3 + 4sin*θ*)). Hence find the equation of the tangent at the point (4, 3+2.
24. Find the equation of the tangents to the following circles at the given points:
25. *x*2 + *y*2 = 5, (-2, 1)
26. *x*2 + *y*2 – 4*x* + 2*y* = 3. (0, -3)
27. *x*2 + *y*2 + 6*y* – 1 = 0, (3, -4)
28. 2*x*2 + 2*y*2 + 9*x* – 4*y* + 4 = 0, (-2, 3)
29. Find the equation of the circle whose centres lies on the line *y* = 3*x* – 7 and which passes through the points (1, 1) and (2, -1).
30. Show that the distance of the centre of the circle *x*2 + *y*2 – 6*x* – 4*y* + 4 = 0 from the *y-*axis is equal to the radius. What does this prove about the *y-*axis and the centre?
31. Prove that the circles *x*2 + *y*2 – 4*x* – 6*y* = 0 and *x*2 + *y*2 – 4*x* – 6*y* = 3 are concentric. Find the radius of the common centre.
32. Find the lengths of the tangents drawn from the following points to the given circles:
33. (6, -1), *x*2 + *y*2 = 12
34. (-1, 3), *x*2 + *y*2 – 8*x* + 4*y* + 19 = 0
35. (4, -2), *x*2 + *y*2 – 10*y* – 4 = 0
36. (3, -4), *x*2 + *y*2 + *x* – 3*y* = 0.
37. If O is the origin and *P*, *Q* are the intersections of the circle *x*2 + *y*2 + 4*x* + 2*y* – 20 = 0 and the straight line

*x* – 7*y* + 20 = 0. Show that *OP* and *OQ* are perpendicular. Find the equation to the circle through *O*, *P* and *Q*.

1. The circle x2 + y2 + 2gx + 2fy + c = 0 passes through the points A(-1, -2), B(1, 2), C(2, 3). Write down three equations which must be satisfied by g, f, c. Solve these equations and write down the equations of the circle *ABC*.
2. Prove that the line *y* = 3*x* – 1 neither cuts nor touches the circle (*x* – 1)2 + (*y* – 1)2 = 9
3. Find the greatest and least distance of a point P from the origin as it moves round the circle
4. *x*2 + *y*2 – 24*x* – 10*y* + 48 = 0
5. *x*2 + *y*2 + 6*x* – 8*y* – 24 = 0
6. A circle which passes through the origin cuts off intercepts of lengths 4 and 6 units on the positive *x* and *y-*axes respectively. Find the equation to the circle and the equations to the tangents to the circle at the points other than the origin where it cuts the axes.
7. A is the point (3, -1) and B is the point (5, 3). Show that the locus of the point P, which moves so that *PA*2 + *PB*2 = 28 is a circle Find its centre and radius.
8. Prove that the line *y* = 2*x* – 3 is a tangent to the circle (*x* – 5)2 + (*y* – 2)2 = 5
9. Find the equation of the circle which has the points

(-7, 3) and (1, 9) at the end of a diameter. Find also the equation of the tangents to the circle which are parallel

1. to the *x-*axis
2. to the *y-*axis
3. The point (*a*, *b*) is the midpoint of a chord of the circle *x*2 + *y*2 = *R*2. Show that the equation to the chord is *ax* + *by* = *a*2 + *b*2.
4. A circle touches the *x-*axis and cuts off a constant length 2*a* from the *y-*axis. Show that the equation to the locus of its centre is a curve *y*2 – *x*2 = *a*2.
5. Find the length of the chord joining the points in which the straight line = 1 meets the circle *x*2 + *y*2 = *R*2.
6. Show that the line 2*x* – 3*y* + 26 = 0 is a tangent to the circle *x*2 + *y*2 – 4*x* + 6*y* – 104 = 0 and find the equation to the diameter through the point of contact.
7. Find the length of the tangent to the circle *x*2 + *y*2 – 4 = 0 from the point (*x*, *y*) and deduce the equation of the locus of P, when it moves so that the length of the tangents to the circle is always equal to the distance of P from the point (1, 0).
8. Prove that the line *x* – *y* – 3 = 0 is a common tangent to the circles *x*2 + *y*2 – 2*x* – 4*y* – 3 = 0 and *x*2 + *y*2 + 4*x* – 7*y* – 13 = 0. What are the coordinates of the point in which it meets the other common tangent?
9. Show that the common chord of the circles *x*2 + *y*2 = 4 and *x*2 + *y*2 – 4*x* – 7*y* – 4 = 0 passes through the origin.
10. Show that the following pair of circle are orthogonal:
11. *x*2 + *y*2 – 6*x* – 8*y* + 9 = 0, *x*2 + *y*2 = 9
12. *x*2 + *y*2 – 4*x* + 2 = 0, *x*2 + *y*2 + 6*y* – 2 = 0
13. *x*2 + *y*2 – 6*y* + 8 = 0, *x*2 + *y*2 – 4*x* + 2*y* – 14 = 0
14. *x*2 + *y*2 + 10*x* – 4*y* – 3=0, *x*2 + *y*2 – 2*x* – 6*y* + 5=0
15. Prove that the line *y =* 2*x* is a tangent to the curve *x*2 + *y*2 – 8*x* – *y* + 5 = 0 and find the coordinates of the point of contact.
16. A and B have coordinates (-3, 0) and (3, 0). Show that the locus of a point P which moves such that *PB* = 2*PA* is a circle with centre (-5, 0) and radius 4.
17. A triangle has vertices (0, 6), (4, 0), (6, 0). Find the equation of the circle through the midpoint of the sides and show that it passes through the origin.
18. Prove that the following pairs of circles touch each other and state whether the contact is external or internal.
19. *x*2 + *y*2 – 2*x* = 0, *x*2 + *y*2 – 8*x* + 12 = 0
20. *x*2 + *y*2 – 2*x* – 2*y* = 18, *x*2 + *y*2 – 14*x* – 8*y* + 60 = 0
21. *x*2 + *y*2 – 12*x* – 2*y* = 12, *x*2 + *y*2 – 4*x* + 4*y* + 4 = 0
22. *x*2 + *y*2 – 4*x* + 2*y* = 8, *x*2 + *y*2 + 6*x* – 13*y* + 22 = 0
23. Prove that the circle *x*2 + *y*2 – 2*x* – 6*y* + 1 = 0 cuts the circle *x*2 + *y*2 – 8*x* – 8*y* + 31 = 0 in two distinct places and find the equation of the common chord.
24. Points A(0, 2) and B(4, -2) lie on the circumference of a given circle. Points C(-3, -3) and D(7, 2) lie outside the circle but the centres of the circle lie on the CD. Find the equation of the circle.
25. Show that the *x*2 + *y*2 + 4*x* – 2*y* – 11 = 0 and *x*2 + *y*2 – 4*x* – 8*y* + 11 = 0 intersect at right angles.
26. Show that the line *x* + 3*y* – 1 = 0 touches the circle *x*2 + *y*2 – 3*x* – 3*y* + 2 = 0
27. Show that the locus of a point which moves such that the square of its distance from the point (3, 4) is proportional to its distance from the line *x* + *y* = 0, one of the locus being the point (1, 2), is a circle and find its centre and radius.

## Answers

1. *x*2 + *y*2 – 2*x* + 4*y* = 0
2. (a) 1, (-2, 3) (b) 2, (1, 2)

(c) , (, 0) (d) , , 2)

1. (a) *x*2 + *y*2 – 6*x* – 4*y* – 3 = 0

(b) *x*2 + *y*2 + 2*x* + 4*y* + 4 = 0

(c) *x*2 + *y*2 = 25

(d) *x*2 + *y*2 – *x* – 2 = 0

1. *x*2 + *y*2 – 2*x* – 2*y* – 3 = 0
2. *x*2 + *y*2 – 4*x* + 6*y* + 4 = 0
3. *x*2 + *y*2 – 4*x* – 22*y* + 115 = 0
4. (a) *k* > 0 (b) *k* = 1

(b) No value of *k* (c) *k* = 0

1. *x* + 5*y* + 2 = 0
2. (4, 6)
3. (*y* – 3)sin*θ* + (*x* – 2)cos*θ* = 4, *y* + *x* = 10 + 3.
4. (a) *y* = 2*x* + 5 (b) *x* + *y* + 3 = 0

(c) *x* + *y* + 3 = 0 (d) *x* + 8*y* = 32.

1. *x*2 + *y*2 – 5*x* – *y* + 4 = 0
2. (2, 3)
3. (a) 5 (b) 7 (c) 6 (d) 
4. *x*2 + *y*2 + 5*x* – 5*y* = 0
5. *x*2 + *y*2 – 16*x* + 8*y* – 5 = 0
6. (a) 24, 2 (b) 12, 2
7. (4, 1), 3
8. *x*2 + *y*2 – 4*x* – 6*y* = 0

2*x* – 3*y* = 8

*y* = 1, *y* = 11

*x* = 2, *x* = -8

1. 
2. 3*x* + 2*y* = 0
3. , 2*x* – 5 = 0
4. (7, 4)
5. (1, 2)
6. *x*2 + *y*2 – 5*x* – *y* = 0
7. 3*x* + *y* = 15
8. *x*2 + *y*2 – 2*x* + 2*y* = 8

## Locus

1. L and M are the feet of perpendiculars from a point P onto the axes. Find the locus of P when it moves so that LM is length 4 units.
2. A variable line through the point (3, 4) cuts the axes at Q and R. and the perpendiculars to the axes at Q and R intersect at P. What is the locus of the point P?
3. A variable joint P lies on the curve *xy* = 12. Q is the midpoint of the line joining P to the origin. Find the locus of Q.
4. P is a variable line on the curve *y* = 2*x*2 + 3 and O is the origin. Q is the point of intersection of OP nearer the origin. Find the locus of Q.
5. A line parallel to the *x-*axis cuts the curve *y*2 = 4*x* at P and the line *x* = -1 at Q. Find the locus of the midpoint of PQ.
6. Find the locus of a point which moves so that the sum of the squares of its distance from the lines *x* + *y* = 0 and *x* – *y* = 0 is 4.
7. A is the point (1, 0), B is the point (2, o) and O is the origin. A point P moves so that the angle BPO is a right angle and Q is the midpoint of AP. What is the locus of Q?
8. A line parallel to the *y-*axis meets the curve *y* = *x*2 at P and the line *y* = *x* + 2 at Q. Find the locus of the midpoint of PQ.

## Answers to Locus Questions

1. *x*2 + *y*2 = 16
2. *xy* = 3*y* + 4*x*
3. *xy* = 3
4. *y* = 6*x*2 + 1
5. *y*2 = 8*x* + 4
6. *x*2 + *y*2 = 4
7. 4*x*2 + 4*y*2 – 8*x* + 3 = 0
8. 2*y* = *x*2 + *x* + 2

# SERIES AND SEQUENCES

**Objectives:**

After reading this book, you should be able to

* Recognize the difference between a sequence and series
* Recognize an arithmetic progression
* Find the nth term of an arithmetic progression
* Find the sum of an arithmetic progression
* Recognize a geometric progression
* Find the nth term of a geometric progression A. P
* Find the sum to infinity of an a G.P with common ratio when
* Understand for sum of series
* Be familiar with standard formula for
* Prove expressions by mathematical induction.
* Apply geometric and arithmetic progression to solve word problems.

**Introduction**

A **sequence** is a set of number stated in a definite order such that each number forming a set can be obtained from the previous one according to some rule.

For example

A **series** is obtained by adding up all the terms of the sequence for example. Suppose we have a sequence

*u1, u2, u3………………………un*

The serie can be obtained from the above sequence. The serie is *u1+u2+u3+…………………………un*

From the examples of sequences above. It is possible to give a formular for the general or nth term of each of the above series**.**

Thus for *a*, *un=2n* *,* where *n =* 1, 2, 3, .…

**For *b*,** where *n =* 1, 2, 3, .…

**For *c***, *un* is not easy to find but we can give a separate formulae for odd and even terms of the sequence.

For *d* where *n =* 1, 2, 3, .…

For *e*,the formula is not obvious

To obtain each successive term we divide by 2 then change sign. Thus at each stage we multiply by which means that

For *f*, each term is a sum of two previous terms. This can be expressed formally as a relation between (*un* = *un* – 1 + *un* – 2). In this case more advanced technique are needed to find the formula.

If the sequence ends after a certain number of terms it’s said to be **finite**. A sequence which continues indefinitely is said to be **infinite**.

## ARITHMETIC PROGRESSION (A.P)

Arithmetic progression is a serie in which one term is obtained from the previous one by adding a constant number.

For example:

46+50

The constant number is called a common difference. In the above examples the common differences are 1 for *a*, 4 for *b* and -3 for *c* respectively.

The arithmetic progression is completely defined when the first term (***a***) and common difference, (***d***) are given

Generally, the arithmetic progression series is given by

**Example 1**

Which of the following series are arithmetic progression. Write down the common difference of those that are

…

***Solution***

Is an arithmetic progression with first term *a* = 7 and common difference *d* = 4

(*common difference*)

It is an arithmetic progression with first term *a* = -7 and common difference 2

It is an arithmetic progression with the first term

*a* = -17 and common difference (*d*) = 5

 is an arithmetic progression

12 + 22 + 32 + 42 + … is not an arithmetic progression.

**Example II**

Write down the terms indicated in each of the following arithmetic progression

***Solution***

2 + 6 + 10 + 14 + … 12th

The *n*th term of an A.P (*un*) is given by:

*a* = 2, *d* = 4

12th term, n=12

*n*th

**Example III**

Find the number of terms in each of the following progression

***Solution***

The *n*th term

**Example IV**

An AP is given by

(i) Find the sixth term

(ii) Find the nth term

(iii) If the 20th term is 15. Find K

***Solution***

But

**Sum of Arithmetic Progression**

Generally, the arithmetic progression is given by

*a* + (*a* + *d*) + (*a* + 2*d*) + … *a* + (*n* – 2)*d* + *a* + (*n*–1)*d*

Let the sum of the arithmetic progression be

……….. (1)

Suppose the terms are added in opposite order

………….. (2)

………….. (3)

Since there are *n* terms of equation (3) that have been added together. The total is

|  |
| --- |
|  |

This is the formula for the sum of arithmetic progression

**Example I**

Find the sum of the first 50 terms of an A.P

***Solution***

***Example II***

Find the sum of to 20 terms

***Solution***

**Example III**

Find the sum 27 + 22 + 17 + … to 10 terms

***Solution***

**Example IV**

Find the sum of the following A.P

1. 6+10+14+…50
2. 10+7+4+…-50
3. 5+9+13+…101
4. 83+80+77+…5

***Solution***

(-3)

= −420

= 193.5

**Example V**

In an A.P the sum of the first 10 terms is 520 and the 7th terms doubles the 3rd term. Find the first term *a* and common difference (*d*).

***Solution***

7th term

*un*

*u*7 *d*

3rd term

*u*3 *=*

*u*7 = 2*u*3

Substituting eqn (2) in eqn (1)

***Example V***

The first and last terms of an A.P with 25 terms are 29 and 179 respectively. Find the sum of the series and it’s common difference

***Solution***

, , nth term

*un*

**Example VI**

The nth term of the series is 10-3n. Find the first three terms of the series and the sum of the first 15 terms.

***Solution***

***Example VIII***

Given that the first and third terms of an A.P are 13 and 25 respectively. Find the 100th term and the sum of the first 15 terms

***Solution***

*u*100

The second and seventh terms of an A.P are -5 and 10 respectively. Find the fifth term and the least number of terms that must be taken for their sum to exceed 200

***Solution***

………………………. (1)

………………………. (2)

*a* = -5 – *d*

*un* = *a* + (*n* – 1)*d*

*u*5 = -8 + (5 – 1) × 3

*u*5 = -8 + 12

*u*5 = 4

*Sn* > 200

> 200

-19*n* + 3*n*2 > 400

3*n*2 – 19*n* – 400 > 0





By completing squares;



**Example VIII**

In an A.P the sum of the first 15 terms is 615 and the 13th term is six times the second term. Find the first three terms

***Solution***

15(*a* + 7*d*) = 615

………………… (1)

*u*13

*u*13 = 6*u*2

………………… (2)

Substituting Eqn (1) in Eqn (2)

*a* = 41 – 7*d*

The first three terms are:

**Example IX**

In an A.P the sum of the first 2n terms is equal to the sum of the next n terms. If the first term is 12 and common difference is 3. Find the non-zero value of n.

***Solution***

In an A.P the sum of the first 2*n* terms

The total of 2*n* terms and the next n terms is 3n terms.

The sum of the next *n* terms after 2*n* terms is

6

Sum of first 2*n* terms = sum of the next *n* terms.

**Example X**

The sum of the first n terms of a certain series is

. Show that the series is an A.P find the first term and common difference.

***Solution***

|  |
| --- |
|  |

Common difference=6

4, 10, 16, …

The series is an A.P

**Example XI**

The sum to n terms of a particular series is given by

(a) Find an expression for the sum of (n-1) terms

(b) Find the expression for the nth term of the series.

Show that the series is arithmetic progression. Find the first term and common difference.

Solution:

Common difference

**Example XII**

The sum of the three terms in A.P is 30 and the sum of their squares is 398. Find the numbers.

***Solution***

………………………. (1)

……….. (2)

Substituting *a* = 10 – *d* in Eqn (2);

The numbers are 3, 10 and 17.

**Example XIII**Show that the sum to 20 terms of the series

can be written in the form and find x and y.

***Solution***



## GEOMETRIC PROGRESSION

A geometric progression G.P is a sequence where each new terms after the first is obtained by multiplying the preceding term by a constant **r**, called a common ratio. If the first term of the sequence is ‘*a*’ then the geometric progression is

Examples of geometric progression are:

From the above examples

First term

For

*a* = 1, common ratio (*r*)

For

First term *a* = 27

**Example I**

Which of the following series are geometric progression write down the common ratios of those that are:

…

…

(iii) 1+1.1+1.21 + 1.331…

(iv) 1+1/4+1/16+1/64+ …

(v) …

***Solution***

(i) 3 + 9 + 27 + 81 …

=3

…is a geometric progression

Common ratio

is a geometric progression with a common ratio

…

It is a geometric progression with a common ratio r=1.1

….

It is a geometric progression with a common ratio

+ …







… is not a geometric progression because the common ratio is not the same.

**Example II**

State the common ratio and the next two term of the following geometric progression (G.P)

***Solution***

Let the next two terms be x and y

And

Let the next numbers of the sequence be m and n.

*n* = 4

Let the next two numbers of the series be m and n





Let the next terms of the sequence be *m* and *n*.

162 + 54 + 18 + *m + n* + …



3*m* = 18

*m* = 6



3*n* = 6

*n* = 2

The next two terms of the sequence

162 + 54 + 18 + *m + n* + … are 6 and 2

**Example III**

Write down the terms indicated in each of the following progression

***Solution***

The *n*th of a geometric progression is given by;

 = 2

For the 8th term,

The *n*th term

For the 5th term

**Example IV**

 The 6th term is 

For the 7th term, *n =* 7

**Example VI**

How many term are in a geometric progression

Solution:

The *n*th term is given by

**Example VII**

The first term of a G.P with positive common ratio is 80. If the sum of the first three term is 185, find the common ratio.

***Solution***

**Example VIII**

In a G.P the second term exceeds the first by 20 and the fourth exceeds the second by 15. Find the two possible values of the first term

***Solution***

……………………. (1)

……………………. (2)

If *r =* , 20 = *ar – a*

When *r* = ,

When *r* = 1, *a* is not defined.

**Example IX**

The second and fifth term in a G.P are 405 and -120 respectively. Find the seventh term.

***Solution***

………………………… (1)

……………………… (2)

Eqn (2) ÷ Eqn (1)

**Example IX**

The second, fourth and eighth terms of an A.P are in geometric progression, the sum of the third and fifth terms is 20. Find the first four terms of the progression.

***Solution***

………………………… (1)

=20

………………………… (2)

Substitute Eqn (1) in Eqn (2)

**Sum of a Geometric Progression**

Let the sum of a geometric progression be

……… (1)

…… …(2)

|  |
| --- |
|  |

**Example I**

Find the sum of six terms of the progression

***Solution***

**Example II**

Find the sum of the first 20 terms of a G.P with first term 3 and common ratio 1.5

***Solution***

**Example III**

Find the sum of the first six terms of a G.P

***Solution***

**Example IV**

Find the sum of the following G.P Progression

1. 

***Solution***







× 4

**(d)**



**Example V**

Find the sum of the first n terms of the G.P  
how many terms of the series are needed to reach a sum greater than 100

***Solution***



**Example VI**

The sum of the first seven term of a G.P is 7 and the sum of next seven terms is 896. Find the common ratio of the progression. If the *k*th term is the first term of a G.P which is greater than 1, find *k*.

***Solution***

Eqn (1) ÷ Eqn (2)



*p* + 1 = 129

*p* = 128

But *p* = *r*7

**Example VII**A G.P has first term 16 and common ratio ¾. If the sum of the first n terms is greater than 60. Find the least possible values of n.

***Solution***



**Sum to Infinity of A G.P**

Consider the general geometric progression (G.P)

The sum of the *n* terms is denoted by

, for |*r*| < 1

Since *n* cannot be negative it implies that as *n* increases decreases for |*r*| < 1

But as *n* tends to a big positive value (*n* →∞),

 (sum to infinity of a G.P)

We say that the G.P with a common ratio, *r* and first term (*a*) converges when and the limit of the sum is . There is no limit of a geometric progression whose common ratio lies outside the range

**Example I**Find the sum to infinity of a G.P with first term 3 and common ratio

***Solution***

**Example II**

Find the sum to infinity of

+ …

***Solution***

**Example III (UNEB Question)**

Find how many terms of the series   
must be taken so that the sum will differ from the sum to infinity by less than

***Solution***

(*Sign changes because the denominator is negative*)

**Example IV**

The sum to infinity of a G.P is twice the sum of the first two terms. Find the possible values of the common ratio.

***Solution***



**Example V**

The first and fourth terms of a geometric series are 135 and −40 respectively. Find its common ratio and the sum to infinity.

***Solution***



**Example VI**

The sum of the first n terms of a geometric progression is Find the first term, its common ratio and its sum to infinity.

***Solution***

Comparison

The sum to infinity = 8

**Example (UNEB Question)**

In an arithmetic progression *u*1 + *u*2 + *u*3 + ...., *u*4 = 15 and u16 = -3. Find the greatest integer *N* such that *uN* ≥ 0. Determine the sum of the first *N* terms of the progression.

***Solution***

Let *a* = 1st term of the A.P. and

*d* = common difference of the A.P.

The *n*th term of the A.; *u*n = *a* + (*n* − 1)*d*

=> 4th term, *u*4 = *a* + 3*d*

But *u*4 = 15

=> *a* + 3*d* = 15.............................. (i)

The 16th term *u*16 = *a* + 15*d*

But *u*16 = -3

=> *a* + 15*d* = -3............................. (ii)

Eqn (i) − Eqn (ii)

-12*d* = 18

*d* = -1.5

From eqn (i)

*a* + 3(-1.5) = 15

*a* = 15 + 4.5

*a* = 19.5

Substituting for *a* and *d* in *un*= a + (*n* − 1)*d*

=> *un* = 19.5 + (*n* − 1) ×-1.5

=19.5 − 1.5(*n* − 1)

Now for *un* ≥ 0

=> 19.5 − 1.5(*n* − 1) ≥ 0

19.5 − 1.5*n* + 1.5 ≥ 0

21 − 1.5*n* ≥ 0

21 ≥ 1.5*n*

14 ≥ *n*

Hence *n* ≤ 14

The greatest integer, *N* is 14

Sum of *n* terms of the A.P. is given by





**Example (UNEB Question)**

Show that ln2*r*, *r* = 1, 2, 3 is an arithmetic progression.

ii) Find the sum of the first 10 terms of the progression.

iii) Determine the least value of m for which the sum of the first 2*m* terms exceeds 883.7.

***Solution***

a) i) For *r* = 1,

ln21 = ln 2

For *r* = 2



Common difference = 2ln2 – ln2 = ln2

For *r* = 3

ln23 = 3 ln2

Common difference = 3 ln 2 – 2 ln 2 = ln2

Since the difference between any two consecutive terms is the same i.e. ln 2, Therefore the progression is an arithmetic progression.

**ii**) For an *A*.*P*., 

Where *n* = 10, *a* = ln2 and *d* = ln2



= 38.1231 (4 dp)

1. Give *a* = ln2

*d* = ln2

*n* = 2m



For,







Hence the least value of m is 25.

**Example (UNEB Question)**

The first term of an arithmetic progression (A.P) is 73 and the 9th is 25. Determine

1. The common difference of the A.P.
2. The number of terms that must be added to give a sum of 96.

**b**) A geometric progression (G.P.) and an arithmetic progression (A.P) have the same first term. The sums of their first, second and third terms are 6, 10.5 and 18 respectively. Calculate the sum of their fifth terms.

***Solution***

Let *a* = 1st term and

*d* = common difference of an A.P

Now 1st term = 73 and

The 9th term = 25

But 9th term = *a* + 8*d*

=> *a* + 8*d* = 25

73 + 8*d* = 25

8*d* = 25 − 73

8*d* = 48

*d* = -6

Hence the common difference of the A.P is -6

ii) Let *n* = number of terms required



But *Sn* = 96



Substituting for *a* = 73 and *d* = -6



73*n* − 3*n*2 + 3*n* = 96

3*n*2 − 76*n* + 96 = 0



Hence the terms that must be added to give a sum of 96 are 24

**b**) Let *a* = 1st term of both A.P. and G.P.

*d* = common difference of A.P. and

*r* = common difference of a G.P.

Now for an A.P:

1st term = a

2nd tern = *a* + *d*

3rd term = *a*+2*d*

4th term = *a* + 3*d*

5th term = *a* + 4*d*

Also for a G.P.:

1st term = *a*

2nd term = *ar*

3rd term = *ar*2

4th term = *ar*3

5th term = *ar*4

Sum of their 1st terms = 2*a*

⟹ 2*a* = 6

*a* = 3

Sum of their second terms = *a* + *d* + *ar*

⟹ *a* + *d* + *ar* = 10.5

3 + *d* + 3*r* = 10.5

*d* + 3*r* = 7.5................................ (i)

Sum of their third terms = *a* + 2*d* + *ar*2

⟹ *a* + 2*d* + *ar*2 = 18

3 + 2*d* + 3*r*2 = 18

2*d* + 3*r*2 = 15

Making *d* the subject from Eqn (ii)

*d* = 7.5 − 3*r*

Substitute for *d* into Eqn (ii)

2(7.5 − 3*r*) + 3*r*2 = 15

15 − 6*r* + 3*r*2 = 15

3*r*2 − 6*r* = 0

*r* = 2

*d* = 7.5 − 3 × 2 = 1.5

The sum of their 5th terms = *a* + 4*d* + *ar*4

= 3 + 4 × 1.5 + 3 × 24

= 3 + 6 + 48

= 57

**Example (UNEB Question)**

a) The *n*th term of a series is Un = *a*3n + *bn* + *c*. given that *U*1 = 4, *U*2 = 13 and *U*3 = 46, find the values of *a*, *b* and *c*.

***Solution***

**a**) Given *Un* = *a*3*n*+ *bn* + *c*

Substituting for *n* = 1, 2, 3...

For *n* = 1

* 3*a* + *b* + *c* = 4 ....................................(i)

For *n* = 2,

* 9*a* + 2*b* + *c* = 13 ................................(ii)

For *n* = 3

* 27*a* + 3*b* + *c* = 46..............................(iii)

Eqn (ii) − Eqn (i)

6*a* + *b* = 9........................................ (iv)

Eqn (iii) − Eqn (ii)

18*a* + *b* = 33..................................... (v)

Eqn (v) − Eqn (iv)

12*a* = 24

*a* = 2

Substitute for *a* in Eqn (iv)

12 + *b* = 9

*b* = -3

Substituting for b in Eqn (i),

6 − 3 + *c* = 4

*c* = 1

 *a* = 2, *b* = -3 and *c* = 1

**Example (UNEB Question)**

**9**.**(a)** The tenth term of an arithmetic progression (A.P) is 29 and the fifteenth term is 44. Find the value of the common difference and the first term. Hence find the sum of the first 60 terms.

**b)** A cable 10 m long is divided into ten pieces whose lengths are in a geometric progression. The length of the longest piece is 8 times the length of the shortest piece. Calculate to the nearest centimeter the length of the third piece.

***Solution***:

**(a)** The nth term of A.P. is given by

*Un* = *a* + (*n* – 1)*d*

Given *U*10 = 29,

⟹ *a* + 9*d* = 29 ……………….……………(i)

Given *U*15 = *a* + 14*d*

⟹ *a* + 14*d* = 44 ……………………..…...(ii)

Eqn (*i*) – Eqn (*ii*);

5*d* = 15

*d* = 3

Substituting for *d* in Eqn (*i*)

*a* + 27 = 29

*a* = 2

The sum of the first *n* terms of A.P. is given by

.



**b**) Given length of cable = 10m

Number of divisions = 10

Here, length of divisions makes a G.P.

By considering the first to be the shortest and the 10th to be the longest,

Let 1st term = *a*

10th term = *ar*9

3rd term = *ar*2

*ar*9 = 8*a*

*r*9 = 8





Length of the 3rd piece, *U*3 = *ar*2

*U*3 = 0.2862 × 2⅔

= 0.4544m

~ 45 cm

**Application of Geometric Progression**

**UNEB Question**

A credit society gives out a compound interest of 4.5% per annum. Mugagga deposited 300,000 at the beginning of each year. How much money will he have at the end of the 4th year in no withdrawal is made between this period.

***Solution***

**Example II (UNEB Question)**

A Finance society gives out a compound interest of per annum. Moses deposited £100 into a saving account at the beginning of each year. How much will his saving be worth after ten years?

Solution:

1564.5548746

**Example III**

5 millions are invested each year at a rate of 15% interest. In how many years will it accumulate to more than 50 millions.

**Example IV**

A man pays a premium of £ at the beginning of every year to insurance company on understanding that at the end of fifteen years he can receive back the premiums which he has paid with 5% compound interest. What should he receive? Give your answer correct to three significant figure.

***Solution***

**Proof by induction of summation of other series**

Mathematical induction is a method of proof in which a statement is proved for one step in a process and it’s shown that if the statement holds for that step, it holds for the next. Proof by induction is not a direct proof because the method doesn’t produce the expression it’s self but it can only be used to prove that a given expression is a required sum.

Mathematic proof by induction involves 3 steps

**Step 1:** (initial step) is to prove the given statement for all natural numbers.

**Step 2:** (induction step) prove the given statement for any natural number implies the given statement for the next natural numbers.

**Step 3:** (conclusion step).

If you have just covered induction with your teacher and you are feeling uneasy about the whole thing. If you are anything like me, you are having the same thought that I had. Can’t you prove anything is true. If you assume it to be true in the first place, what is left is to prove.

If you have already proved isn’t that cheating any how

Well………

First step don’t get mad at your teacher.

Induction makes a perfect sense to him and he honestly thinks he has explained it clearly. When I was still at school like you I had a good math teacher so I had good lessons of induction from my teacher.

But still I didn’t trust induction so I tried to pick up where my teacher had left me and fill in some gaps.

Before the lesson ended. I asked my teacher, ‘*excuse me teacher*’

Excuse me teacher, ‘*excuse me teacher, excuse me teacher. Mathematical proof by induction is confusing me because I cannot seem to reason with myself as to how to go about getting to the solution and also asked why is it called mathematical proof by induction not substitution induction*’.

**Example 1**

Prove by induction that

***Solution***

It is true for *n* = 1

Assume the result holds for some general value of n say *n* = *k*.

It must also be true for the next value of n i.e

Adding the next term,

= (k + 1)[*k* + 1]

1 + 2 + 3 + …+*k*+*k* + 1 = (*k* + 1)(*k* + 2)

For the next integer

Since it’s true for *n* = 1, *n* = 2 and so on hence it’s true for all positive integral values of n.

**Example II**

Prove by induction that

***Solution***

12

It’s true for *n* = 1.

Assume the result holds for the general value of n say n=k, it must be true for the next value of n i.e

Adding the next term

For the next integer

Since it is true for *n* = 1 and *n* = 2 and so on, and so on hence it’s true for all positive integral values of n.

**Example III**

Prove by induction that

***Solution***

For

It’s true for

Assume the result holds for the general value of n say

It must be true for the next integer

Adding the next term



For the next integer

Since it’s true for so on hence it’s true.

**Example IV**

Prove by induction that

***Solution***

For

It’s true for

Assume the result holds for general value of *n*,say

It must be true for the next integer

adding the next term

For the next integer,

Since it is true for and so on hence it’s true for all positive integral value of n.

**Example V**

Prove by induction that

***Solution***

It’s true for

Assume the result holds for the general value of

It must be true for next integer

Adding the next term



13 + 23 + 33 + … *k*3 + (*k* + 1)3 = 

Since it true for *n* = 1, *n* = 2 and so on, it is true for all positive integral value of n.

**Example** **VI**

Prove by induction that

***Solution***

For

Its true result holds for general value of n say

It must be true for the next integer



For the next integer, ,

8 = 8

Since it is true for and so on hence it’s true for all positive integral values of n.

**Example VII**

Prove by induction that

***Solution***

For

Assume the result holds for the general value of *n* say

*n = k*.

It must be true for the next integer

For the next integer,

Since it is true for , and so on, it is true for positive integral values of *n*.

**Example VIII**

Prove by induction.

***Solution***

For

It is true for *n* = 1.

Assume the results holds for the general value of

It must be true for the next integer

Adding the next term



For the next integer

Since it is true for and so on hence it is true for all positive integral values of n.

**Example IX**

Prove by induction that is divisible by 5 for all positive integral value of n

***Solution***

It’s true for *n* = 1

Assume it holds for the general value of n say

………………… (i)

It must be true for the next integer

……………..…. (ii)

= 

For the next integer

Since it is true for *n* = 1, n = 2 and so on, it is true for all the positive integral values of *n*.

**Example X**

Prove by induction that is divisible by 3 for all positive integral value of n.

***Solution***

It is true for *n* = 1

Assume the result holds for general value of n say *n= k*

…………………… (i)

It must be true for the next integer

…………….….. (ii)

For the next integer,

Since it’s true for and so on it’s true for all positive integral values of n.

**Example**

Prove by induction that 8*n* – 7*n* + 6 is divisible by 7 for all positive values of n.

***Solution***

For

It’s true for for the general value of n say

………………….. (1)

It must be true for the next integer

………………….. (2)

For the next integer

Since it is true for and so on it’s true for all positive integral values of *n*.

**Example (UNEB Question)**

Prove by induction that is always divisible by 7 for

***Solution***

Assume the result holds for the general value of n say

………………….. (1)

For the next integer,

………………. (2)

Eqn (2) – Eqn (1)

=

For the next integer ,

Since it is true for and so on it’s true for all positive of *n*.

**Example XI**

Prove by induction that 8*n* – 6*n* is always divisible by 7 for all even integers of *n*.

***Solution***



For *n* = 2, 

Assume the result holds for the general even integer *n = k*



7*ak* = 8*k* – 6*k*. ……………… (1)

For the next even integer, 

7*ak* + 1 = 8*k* + 1 – 6*k* + 1 ………... (2)

Eqn (2) – Eqn (1)

7(*ak* + 2 – *ak*) = 8*k* + 2 – 6*k* + 2 – 8*k* + 6*k*

7(*ak* + 2 – *ak*) = 8(82 – 1) – 6*k*(62 – 1)

= 8*k*(63) – 35(6*k*)

*ak* + 2 – *ak* = 9(8*k*) –5(6*k*)

*ak* + 2 = *ak* + 9(8*k*) – 5(6*k*)

For the next integer *n* = *k* + 2 = 4, *k* = 2

*a*4 = *a*2 + 9(82) – 5(62)

*a*4 = 4 + 396

*a*4 = 400

From *an* = , = 400

Since it is true for *n* = 2, *n* = 4, and so on, hence it is true for all positive even integers.

## Revision Exercise

1. Which of the following series are arithmetical progressions? Write down the common difference of those that are.
2. 7 +  + 10 +  + …
3. -2 – 5 – 8 – 11 + …
4. 1 + 1.1 + 1.2 + 1.3 + …
5. 1 + 1.1 + 1.1 + 1.11 + …
6. , +  +  +  + …
7. 12 + 22 + 32 + 42 + …
8. *n* + 2*n* + 3*n* + 4*n* + …
9. 1 +  +  +  + …
10. Write down the first term and common difference of each of the following arithmetic progressions.
11. 8 + 11 + 14 + 17 + …
12. 23 + 25 + 27 + 291 + …
13. 19 + 16 + 13 + 10 + …
14.  + 15 +  + 18 + …
15. Find the sums of the following arithmetic progressions
16. 4 + 11 + … to 16th term
17. 3 +  + … to 20th term
18. 19 + 13 + … to 10th term.
19. -9 – 1 + … to 8th term.
20. Write down the terms indicated in each of the following Arithmetic progressions.
21. 3 + 11 + …, 10th and 19th
22. 8 + 5 + …, 15th, 31st term.
23.  12th, *n*th
24. 50 + 48 + … , 100th, *n*th
25. Find the 18th term of a series that has an *n*th term given by (2 + 3*n*)
26. Find the sum of the arithmetic progression -7 – 3 + 1 + … from the 7th to the 13th term inclusive.
27. Find the number of terms in the following arithmetic progressions:
28. 2 + 4 + 6 + … + 46
29. 50 + 47 + 44 + … + 14
30. 2 + 4 + … + 4*n*
31. *a* + (*a* + *d*) + … + *l*
32. Find the 31st term of a series that has an *n*th term given by 
33. Find the sum of all odd numbers between 0 and 500 which are divisible by 7.
34. The 2nd term of an arithmetic progression is 15 and the 15th is 21. Find the common difference, the first term and the sum of the first 10 terms.
35. Find the 5th and 7th terms of a series that has an *n*th term given by (-1)*n*(2*n* + 1)
36. Show that the sum 1 + 3 + 5 + … + (2*n* – 1) is always a perfect square.
37. The 4th term of an arithmetic progression is 18, and the common difference is -5. Find the 1st term and the sum of the 1st 16 terms.
38. Find an expression for the *n*th term of each of the following arithmetic progression and use your answer to write down the 100th term of each series.
39. 5 + 8 + 11 + 14 + …
40. 5 + 2 + -1 – 4 + …
41. 12½ + 16 + 19½ + 23 + …
42. A piece of string of length 5m is cut into *n* pieces in such way that the lengths of the piece are in an arithmetic progression. If the lengths of the longest and shortest pieces are 1m and 25 cm respectively, calculate *n*.
43. Find the difference between the sums of the first 10 terms of the A.P whose first terms are 12 and 8 and whose common differences are respectively 2 and 3.
44. Find the sum of each of the following arithmetic progressions:
45. 2 + 4 + 6 + 8 + 10 + … 146
46. 100 + 95 + 90 + 85 + 80 + … + 20
47. 4 + 10 + 16 + 22 + 28 + … +334
48. 5¼ + 4½ + 3¾ + … + -3
49. The 10th term of an arithmetic progression is 10 and the sum of the first 10 terms is -35. Find the first term and the common difference of the progression.
50. The first term of an arithmetic progression is -12, and the last term is 40. If the sum is 196, find the number of terms and the common difference.
51. In an arithmetic progression, *u*5 = -0.5 and *S*7 = 21. Find *a*, *d*, and *u*6.
52. The sum of the first four terms of an arithmetic progression is twice the 5th term. Show that the common difference is equal to the first term.
53. Find the sum of the even number divisible by 3 lying between 400 and 500.
54. The sum of the first ten terms of an arithmetic progression is 120 and the sum of the first 20 terms is 840. Find the sum of the first 30 terms.
55. Find the arithmetic mean of:
56. 3 and 27
57. 3 and -27
58.  and 
59. log 3 and log 27
60. Show that the sum of the integers from 1 to *n* is .
61. An arithmetic progression has a common difference, *d*. If the sum to 20 terms is 25 times the first term, find in terms of *d*, the sum to 30 terms.
62. Three numbers in an arithmetic progression have sum 33 and product 1232. Find the numbers.
63. Show that the sum of the first *n* terms of the arithmetic progression with first term *a* and common difference *d* is .
64. In an arithmetic progression, *a* = -61 and *d* = 4. Find the least value of *n* such that *Sn* > 0.
65. An arithmetic progression has first term -5 and common difference 1.5. Find the greatest number of terms the arithmetic progression can have given that the sum of the terms does not exceed 450.

## Answers

1. (a) 1½ (b) -3 (c) 0.1 (d)  (g) *n*

2. (a) 8.3 (b) 23, 2 (c) 19, -3 (d) 13½, 1½

3. (a) 904 (b) 1188 (c) 88 (d) 193½

4. (a) 75, 147 (b) -34, -82 (c) , 

(d) -148, (52 – 2*n*)

5. 56 6. 1512

7. (a) 23 (b) 13 (c) 2*n* (d) 

8. 24 9. 9072 10. 2, 13, 220

11. 13, -15 13. 33, -72

14. (a) 2 + 3*n*, 302

(b) 8 – 3*n*, -292

(c) ½(18 + 7*n*), 359

15. 8 16. 5

17. (a) 5402 (b) 1000

(c) 9464 (d) 13½

18. -17, 3 19. 14, 4

20. 13½, -3½, -14½

22. 7650 23. 2160

24. (a) 15 (b) -15 (c)  (d) log 9

26. 1575*d* 27. 8, 11, 14

29. 32 30. 28.

## Revision Exercise two

1. For each of the following G.Ps, state the common ratio and the next two terms.

(a) 4 + 20 + 100 + 500 + …

(b) 24 + 12 + 6 + …

(c) 45 + 15 + 5 + …

1. Write down the terms indicated in each of the following G.Ps. Do not simplify your answer.

(a) 5 + 10 + …, 11th, 20th

(b) 10 + 25 + …, 7th, 19th

(c) , 12th, *n*th

(d) 3 – 2 + …, 8th, *n*th

1. Find the sum of the following G.Ps

(a) 100 + 10 + … to 7 terms

(b) 1 –  + … to 6 terms

(c) 3 – 6 + … to *n* terms

(d) *ap* + *ap*+3 + *ap+*6 + … to *k* terms

1. Using the formula , find *S*5 and *S*6 for the G.P 18, -9, 4½, … and hence deduce the value of *U*6.
2. Find the number of terms in the following geometric progressions:

(a) 81 + 27 + 9 + … + 1/27

(b) 0.03 + 0.06 + 0.12 + … 

(c) 

(d) 5 + 10 + 20 + … + 5 × 2*n*

(e) *a* + *ar* + *ar*2 + … + *arn* – 1

1. Find the distinct numbers *p* and *q* such that *p*, *q*, 10 are in arithmetic progression and *p*, *q*, 10 are in geometric progression.
2. Find the value of the common ratio of the G.P that has a third term equal to 6and 8th term equal to 1458.
3. The third term of a geometric progression is 10 and the 6th term is 80. Find the common ratio, the first term and the sum of the first six terms.
4. Find the geometric mean of:

(a) 3 and 27

(b)  and 

(c) 103 and 1027

1. In a geometric progression, the 7th term equals 8 and the 9th term equals 18. Find the possible values of the common ratio.
2. The third term of a G.P is 2 and the fifth is 18. Find two possible values of the common ratio and the second term in each case.
3. Given that the geometric mean of the numbers 4*x* – 3 and 9*x* + 4 is 6*x* – 1, find the value of *x*.
4. Find the sum of the first ten terms of a G.P that has a sixth term  and a seventh term of .
5. The three numbers *n* – 1, *n*, *n* + 3 are consecutive terms of a geometric progression. Find *n* and the term after *n* + 3.
6. If the sum of the first two terms of a G.P is 162 and the sum of its first four terms is 180, find the sum of the first 6 terms, Find also the possible values of the sixth term..
7. The geometric mean of two numbers *a* and *b* (*b* > *a*) is equal to four fifth of the arithmetic mean of the two numbers. If *a* = 6, find the value of *b*.
8. A man starts saving on 1st April. He saves 1p the first day, 2p the second, 4p the third and so on. If he managed to keep on saving under this system until the end of the month (30 days), how much would he have saved?
9. The sum of the first six terms of a G.P is nine times the sum of the first 3 terms. Find the common ratio.
10. Prove that the arithmetic mean of two different numbers exceeds the geometric mean of the same two numbers.
11. Show that the sum of the series 4 + 12 + 36 + 108 + … to 20 terms is greater than 3 × 109.
12. The sum of (*n* + 12) terms of the G.P 2 + 4 + 8 + … is twice the sum of *n* terms of the G.P 2 + 12 + 48 + …. Calculate the value of *n*.
13. A geometric series has first term 5 and common ratio 3. Find the least number of terms the series can have if its series exceeds 2000.
14. Find the ratio of the sum of the first 10 terms of the series log *x* + log *x*2 + log *x*4 + log *x*8 + … to the first term.
15. A geometric series has first term 35 and common ratio 2*x*. State the set of values of *x* for which the series is convergent. Find the value of *x* for which the sum to infinity of the series is 40.
16. The sum of the first two terms of a G.P is 9 and the sum to infinity of the G.P is 25. If the G.P has a positive common ratio *r*, find *r* and the first term.
17. The 2nd, 4th, 8th terms of an A.P are in geometric progression and the sum of the third and fifth terms is 20. Find the first four terms of the progression.
18. For each of the following geometric series, find the range of values of *x* for which the sum to infinity of the series exists.

(a) *x* + *x*2 + *x*3 + *x*4 + …

(b) + …

(c)  + …

(d) …

1. *S* is the sum of *n* terms of a geometric progression, *P* is the product of the *n* terms and *R* is the sum of the reciprocals of the terms. Prove that .
2. Three unequal numbers *a*, *b*, *c* are such that  are in arithmetical progression. Prove that *b*, *a*, *c* are in arithmetical progression.
3. Prove that the G.P  + … is convergent for all values of *x* and find the limit of its sum.
4. Prove by induction that:
5. is divisible by 3
6. is divisible by 3.
7. is a multiple of 21.
8. is divisible by 133.

## Answers

1. (a) 5, 2500, 12500

(b) 

(c) 

2. (a) 5 × 210, 5 × 219

(b) 

(c) 

3. (a) 111.1111

(b) 

(c) 1 – (-2)*n*

(d) 

4. 

5. (a) 8 (b) 7 (c) 8 (d) *n* + 1 (e) *n*

6. -5, 2½

7. 3 8. 2, 2½, 167½

9. (a) 9 (b)  (c) 105

10. ±1½ 11. ±3, ±

12. 13 13. 31

14. 6, 13½ 15. 182, ½, -1

16. 24 17. £1,070,000

18. 2 21. 12

22. 7 23. 1023

24. *x* < 0 25. , 5

26. 2½, 5, 7½, 10

27. (a) |*x*| < 1 (b) |*x*| < 3

(c) |*x*| > 1 (d) 

# COMPLEX NUMBERS

A complex number is represented by an expression of the form where a and b are real numbers and **i** is a symbol with a property .

was introduced by a Swiss mathematician Euler. Traditionally the letters Z and W are used to stand for complex numbers.

Given a complex numbers .

The real part of a complex number *z* is and the imaginary part of z is .

Both and real numbers.

Thus the real part of ***Z*** is and imaginary part of ***Z*** is

By identifying the real number ***a*** with a complex number we consider ℝ (real numbers) to be subset of (complex numbers).

Consider the equation this can be written as and we can see that the equation has no real roots since we cannot find the real root of a negative number, But with (Euler) we are able to find the square root of complex numbers.

**Example**

Solve the following equations

**(a)**

**(b)**

***Solution***

**(b)**

From, 

With this new concept we are in position to find the roots of any quadratic equation.

When the imaginary part of a complex number is zero, the complex number becomes a real number. Thus, all real numbers are complex numbers.

**Definition**

Given a complex number the complex conjugate of **Z** denoted by is a complex number given by = *x* – **i***y*. Therefore if **,**

Then = 4 – 3**i**, -2 – 4**i**

## Algebra of complex numbers

**1. Addition**

Given that two complex numbers  
Then

Therefore if and

**Example**

**1. Subtraction:**

****

Find

**2. Multiplication**

**Example**

,

Find

***Solution***

Find

**3. Division**

**Example I**

Simplify

**Example II**

Express in the form *a* + *b***i**

***Solution***

## The Argand Diagram

Complex numbers can be represented graphically on a graph of Real (Re) and Imaginary (Im) axes called a **complex plane**. The complex plane is similar to the Cartesian plane where the imaginary axis corresponds to the *y*-axis and the real axis corresponds to the x-axis. The diagram representing the complex number in complex plane is called an **argand diagram** named after JR argand 1806.

On the argand diagram a complex number is represented by a line with an arrow on the head to show direction

If we can represent z on argand diagram as shown below.

Im Axis   
 y

If and then

Im axis

Real axis

*y*2

is represented by vectors respectively. The diagram shows that is equal in magnitude and direction

Thus the sum of two complex numbers and is represented in the argand diagram by the sum of the corresponding vectors

**Representing on the argand diagram.**

(z1 – z2) = OP1 – OP2

Since

can be represented by

**Example**

Represent the following complex numbers on the argand diagram.

, ,

***Solution***

0 1 2 3 4 5

Real axis

Im

axis

−5 −4 −3 −2 −1 5

1

2

3

4

−1

−2

−3

−4

*z*1

*z*4

*z*2

*z*3

*z*5

## Modulus of a complex number

Given a complex number , the magnitude or length of z is denoted be is defined by

**Example I**

Given find

**Solution**

**Example II**

Find

***Solution***



= 1

**Example III**

find

***Solution***

**Properties of modulus**

If are complex numbers then



**Example I**

and

Find

***Solution***

,

|(5 – 12i)(3 – 4i)|

= |5 – 12*i*||3 – 4*i*|

***Alternatively***

2

***Alternatively***, 

**Argument of a complex number *Z* (arg Z)**

The argument of a complex number z is defined to be the angle (*θ*) which the complex number *z* makes with the positive *x*-axis.

*x*

*y*

From the diagram above,

**Note:** For a given complex number, there will be infinitely many possible values of the argument, any two of which will differ by a whole multiple of .

To avoid confusion we usually work with the value of for which or . This is called the principle argument of z denoted by **arg z**.

In practice the formula

Which is often used to find the principal argument of a complex number z, despite the fact that it tends to two possible values for in the permitted range. The formula is necessary but not sufficient to help us obtain the arg z. The correct value of arg z is chosen with the aid of a sketch.

**Example**

Find the principal argument of the following complex number

***Solution***

Consider *z*1 = 1 + *i*

*z*1

Since 180° radians



Let

*z*2

-1

**OR**

*−5*

or

**(d)** Let

, from the sketch above

**(e)**

**Properties of Arguments**

|  |
| --- |
| Given the two complex numbers and then |

**Example** I

Given that and . Find the

***Solution***

 = 60°

Z2 = 1 + **i**

1

1

*𝜃*

**Modulus–argument form of a complex number**

(*Polar form of a complex number*)

(x)

Consider a complex number making an angle with the positive

From the diagram above

(modulus argument form a complex number)

Where

**Example**

Express the following complex numbers in modulus –argument

***Solutions***

**(b)**

**(c)**

= 1

-0.5



**(d)** i

**(e)**

z6 = 5(cos 53.1° + i sin 53.1°)

**(g)**

z7 = -5 – 12i

**Example II**

Show that

And

***Solution***







(as required)

**Example III**

Given that

Find in polar form

***Solution***

## Demoivre’s Theorem

Demoivres theorem states that for real values of n

Proving Demoivre’s theorem by mathematical induction

For *n* =1,

It’s true for *n* =1

Assume the results holds for the general value of *n*=*k*

It must be true for the next integer





For the next integer ,

Since it’s true for *n*=1, *n* = 2 and so on it’s true for all positive integral values of n.

**Example I**

Find the value of

***Solution***

******

**Example II**

Express in the form

***Solution***

Let

z = *r*(cos θ + *i*sinθ)



**Example III**

Evaluate

***Solution***:

Let

**Example IV**

Express in modulus –argument form. Hence find

***Solution***

Let



**Example V**

Express (-1+i) in modulus – argument form. Hence show that is real and that

***Solution***

So it is purely real

As required



is purely imaginary.

**Example VI**

***Solutions***

**(a)**

= 

**(b)**

**(c)**

**(d) **

****

****

**(e) **









**(f)**

**Example VII**

Use De-moivre’s theorem to show that

***Solution***

Equating real to real and imaginary to imaginary;

Eqn (1) ÷ Eqn (2)

**Example VIII**

Use Demovre’s theorem to show that

***Solution***

Equating real to real and imaginary to imaginary

Eqn (ii) ÷ Eqn (i)

**Example IX**

Show that

Hence show that 

***Solution***

But 

16cos4*θ* = 2cos4*θ* + 4(2cos2*θ*) + 6

**Example XI**

Given that show that   
Hence or otherwise show that

***Solution***



**Example XII**

Prove that

***Solution***

But

But

Eqn (2) – Eqn (1)

## Solving Complex Equations

Given that *x* and *y*real numbers. Find the values of *x* and *y* which satisfy the equation.

***Solution***

×

Equating real to real and imaginary to imaginary

From equation (1)

From Eqn (2), 

4*x*2 + 4 = 2*xy* + *y*2

For *y* = 0, 4*x*2 + 4 = 0

*x*2 + 1 = 0

*x*2 = -1

*x*2 = *i*2

*x* = ±*i*

For *xy* = 2, 





Let *x*2 = *m*



4*m*2 – 4 = 0

*m*2 – 1 = 0

(*m* + 1)(*m* – 1) = 0

*m* = 1, *m* = -1

When *m =* 1, *x*2 = 1  *x* = ±1

When *m* = -1, *x*2 = *i*2 *x* = ±*i*

*xy* = 2

If *x* = 1, *y* = 2

If *x* = -1, *y* = -2

If *x* = *i*, *y* = -2*i*

If *x* = -*i* , *y* = 2*i*

**Example II**

Find the values of *x* and *y* in

***Solution***

Similarly,

Solving eqn (1) and eqn (2) simultaneously

**Example III**

Find the values of *x* and *y* if

***Solution***

Equating real to real and imaginary to imaginary;

…………… (1)

…………… (2)

Solving Eqn (1) and Eqn (2) simultaneously;

**Example IV**

Find the values of x and y. given that

***Solution***

From equation (1)

From eqn (2)

………(3)

When *x* = 0,

When *xy* = 3

Substituting *y* =  and *xy* = 3 in Eqn (3);



When

When

**Example V**

If *z* is a complex number such that . Where *p* and *q* are real. If |z| = 7, arg P = . Find the value of *p* and *q*.

***Solution***



…………… (1)

Substituting *q* = -4*p* in Eqn (1)



**Example VI**

Given that (1 + *5i*)*p* – 2*q* = 3 + 7*i*, (a) When *p* and *q* are real  
(b) When *p* and *q* are conjugate complex numbers

***Solution***

**(a)**

……………………. (1)

………………………… (2)

From Eqn (2),



**(b)** Let

........................ (1)

.......................... (2)

Substituting Eqn (1) in Eqn (2)

### Square root of Complex Numbers

**Example I**

Find the square root of

***Solution***

Let

Let

But is real

When

**Example VIII**

Find the square root of ***solution***

Let

Equating real to real and imaginary to imaginary;

…………………….(1)

…………………….(2)

Substituting Eqn (2) in Eqn (1)

**Example IX**

Find the roots of

***Solution***

******



But 

16 – 30*i* = *a*2 + 2*ab*i – *b*2

*a*2 – *b*2 = 16

2*ab* = -30

*ab* = -15

****

****

****

Let *m* = *b*2

****

*m*2 + 16*m* – 225 = 0

*m* = 9, *m* = -25

*b*2 = 9

*b* = ±3

*ab* = -15

*a* = 5

When *b* = -3, *a* = 5

When *b* = 3, *a* = -5

*a* + *bi* = 5 – 3*i*, -5 + 3*i*



**Example X**

Show that is a root of the equation

***Solution***

2

2*z*3 – *z*2 + 4*z* + 15



is a root of the equation.

Since is a root of the equation

The complex conjugate must also be a root of the above equation

is also a root of the equation

Sum of roots

Product of roots =

 is a factor of

**Example XI**

Given that is a root of the equation

. Find the other roots

***Solution***

is a root is also a root of the equation

Sum of roots

Product of roots = (2 + 3i)(2 – 3i)

=

= 4 + 9

= 13

is a factor of

are roots of equation of

**Example XII**

Show that is a root of the equation   
 Hence find other roots

***Solution***

*z*2 = 1 + 2*i* + *i*2

is a root of the equation

is also a root of the equation

Sum of the roots

Product of roots = = 2

is a factor of

10

*z*2 – 2*z* + 2 = 0

For z2 + 2z + 5 = 0, 

-1 + 2*i*, -1 – 2*i*, 1 + *i*,are roots of the equation = 0

**Example XIII**

Show that is a root of the equation   
 Find the other roots.

***Solution***

Since is a of the equation and it implies that 1 + *i* is also a root.

Sum of roots

Product of the roots = (1 + *i*)(1 – *i*)

= 2

is a factor of

z2 – 2z + 2

z2 – 2z + 2

**Example XIV**

Given that *z* = 2 – *i* is a root of the equation   
*k* is real. Find other roots.

***Solution***

Sum of roots = 4

Product of roots = 5

is a factor of

are roots of the equation z3 – 3z2 + z + *k* = 0 where *k* = 5

**Example XIV**

Solve for in the simultaneous equations below

***Solution***

From eqn (1)

**Example XV**

Solve the equation *z* 3 – 1

***Solution***

Since

**Alternatively** we can use Demovre’s theorem



(Depending on the number of roots you want)

For *n* = 0,

For *n* = 1, 

For *n* = 2, 

**Example XVI**

Solve:

***Solution***

From

*z* = 

***Alternatively***, we can use Demovre’s theorem

When *n* = 0,

For *n =* 1,

*z* = -3

For *n =* 2,

and

**Example XVII**

Solve the equation

,

For *n* = 1

For *n* = 2,

For *n* = 3

,

**Example XVIII**

Find the fourth roots of -16

***Solution***

******

Let

For *n*  = 2,

For *n* = 3,

**Example XIX**

Find the cube roots of 27i

*i*

## Loci in the complex plane

What is a locus

A locus is a path possible position of a variable point, that obeys a given condition. It can be given as Cartesian equation or it can be described in words.

**Example I**

The complex number z is represented by the point P on the Argand diagram.

Given that find in the simplest form the Cartesian equation of the locus

***Solution***

The locus is a straight line with a positive gradient which can be represented on the complex plane.

**Example II**

Given that . Show that the locus of P is a circle.

***Solution***

Let

This is sufficient to justify that locus is a circle.  
Comparing With



**Example III**

Show the region represented by

***Solution***

Let

It’s a circle with centre (2, -1) and radius less than 1.  
It can be illustrated on the argand diagram

In order to represent on the diagram, we can either take a point inside the circle or outside the circle as our test point.

Taking (2,-1) as the test point.

(the point inside the circle satisfies our locus). It implies that (2,-1) lies in the wanted region. Therefore, we shade the region outside the circle.

**Example IV**

Given that   
find the Cartesian equation of the locus of z and represent the locus by the sketch on the argand diagram. Shade the region for which the inequalities.

***Solution***

The locus is a circle comparing

Center 

For

***Example V***

Shade the region represented by

***Solution***

**Note:** Shade the region represented by |*z* – 1 – *i*| < 3. Implies that we shade the wanted region.

Let

It is a circle with centre (1, 1) and radius less than 9

Taking (1, 1) as our test point

< 9

The region inside the circle is the wanted region.

**Example VI**

Show that when   
the point P(x, y) lies on a circle with centre and radius

***Solution***

Re





Comparing with

*x*2 + *y*2 + 2*x* + *y* = 0 with

, *g* = 1

2*fy* = *y*

**Example VII**

Given that where x and y are real. Show that Imis equation of a straight line

***Solution***

Im

Im = 0

Which is a straight line with a negative gradient.

**Loci in and diagram for arguments of complex numbers**

If is the equation of half line with end point A inclined at an angle to the real axis

A

**Example I**

Sketch the loci defined by the equation

***Solution***

Thus if A is a point representing

is the angle AP makes with the positive real axis. Hence the equation arg represents the half line with end point (1, 2) inclined at angle to the real axis.

**Example II**

Sketch the locus of the equation.

***Solution***

Thus *A* is a point (-2, 0). Arg(z-2) is the angle AP makes with the real axis. Hence represents a half line with end point (-2, 0) inclined at angle measured clockwise from the positive axis.

**Example III**

Show by shading the region represented by

***Solution***

The equations represent half lines with end point (2, 0). Hence the inequality

Represent the two lines and region between them

**Example IV**

Sketch the separate argand diagram the loci defined by

***Solution***

Thus A is a point (-1, 3)

Arg(z – (-1 + 3*i*) is the angle AP makes with the real axis Hence   
is equation of the half line with end point (-1, 3) inclined at an angle of measured clockwise from the real axis

A(-1, 3)

P

Thus, point A is .arg(*z −* (-2 – *i*))is the angle AP makes with the real axis and is the equation of the line through A inclined at and angle of to the real axis

**Sketching of loci involving**

Equation involving are more difficult to interprete. If if necessary

Thus is the angle which the vector AP makes with the vector BP.

If the turn from BP to AP is anti-clockwise the is negative

For instance, if then the locus of P is a circular arc with end point A(3, 0) and (1, 0) such that

Similarly if then the locus of P is a circular arc with end points A (-2, 0) and B(0, +1) such that since both cases the given arguments are positive, the arcs must be drawn so that the turn from BP to AP is anti-clockwise.

P

B(0, 1)

Real axis

Im axis

A(-2, 0)

**Example II**

Sketch on different argand diagram the loci defined by the equations.

***Solution***

The locus of P is a circular arc with end point A(1, 0) and B(-1, 0) such that

P(*x*, *y*)

A(3, 0)

Real axis

Im axis

B(-1, 0)

The locus of P is a circular arc with end points (3, 0) (0, 2) such that

P

A(0, 2)

B(3, 0)

Real axis

Im axis

P

B(4, -2)

A(0, 0)

Real axis

Im axis

**Example**

Find the locus of

**Solution**



which is a circle.

## Revision Exercise 1

1. Prove that if |*Z*| = **r**, then *ZZ\** = **r**2.
2. Express  + *i* in modulus-argument form. Hence find  and  in the form *a* + *ib*.
3. Express -1 + *i* in modulus-argument form. Hence show that (-1 + *i*)16 is real and that  is purely imaginary, giving the value of each.
4. Simplify the following expression:

(a)  (b) 

1. Find the expressions for cos 3θ in terms of cos θ,

sin 3θ in terms of sin θ and tan 3θ in terms of tan θ.

1. Express sin 5θ and cos 5θ/cos θ in terms of sin θ.
2. Prove that . By considering the equation tan 5θ = 0, show that tan2(π/5) = 5 – 2
3. Find expressions for cos 6θ/sinθ in terms of cos θ and for tan 6θ in terms of tan θ.
4. Express in terms of cosines of multiples of θ:

(a) cos5θ (b) cos7θ (c) cos4θ

1. Express in terms of sines of multiples of θ:

(a) sin3θ (b) sin7θ (c) cos4θsin3θ

1. Prove that cos6θ + sin6θ = (3cos4θ + 5)
2. Evaluate (a)  *d𝜃* (b) 
3. (a) Express the following complex numbers in a form having a real denominator.

, 

(b) Find the modulus and principal arguments of each of the complex numbers *Z* = 1 + 2i and *W* = 2 – I, and represent *Z* and *W* clearly by points *A* and *B* in an Argand diagram. Find also the sum and product of *Z* and *W* and mark the corresponding points *C* and *D* in your diagram.

1. If the complex number *x* + *iy* is denoted by *Z*, then the complex conjugate number *x* – *iy* is denoted by *Z\**,

(a) Express |*Z\**| and (*Z\**) in terms of |*Z*| and arg(Z).

(b) If *a*, *b*, and *c* are real numbers, prove that if

*aZ*2 + *bZ* + *c* = 0, then then *a*(Z\*)2 + *b*(Z\*) + *c* = 0

(c) If *p* and *q* are complex numbers and *q* ≠ 0, prove 

1. Find the values of *a* and *b* such that (a + i*b*)2 = *i*. Hence or otherwise solve the equation *z*2 + 2*z* + 1 – *i* = 0, giving your answer in the form *p* + *iq*, where *p* and *q* are real numbers.
2. If , write down the modulus and argument for each of the numbers *Z*, *Z*2, *Z*3, *Z*4. Hence or otherwise, show in the Argand diagram, the points representing the number 1 + *Z* + *Z*2 + *Z*3 + *Z*4.
3. If *Z* = 3 – 4*i*, find

(i) *Z\**  (ii) *ZZ\** (iii) (*ZZ*)\*

1. Simplify each of the following:

(a) (3 + 4i) + (2 + 3i) (b) (2 – 4i) – 3(5 – 3i)

(b) (*2i*)2 (c) i4

1. Simplify each of the following:

(a) (2 + i)(3 – i) (b) (5 – 2i)(6 + i)

(c) (4 – 3i)(1 – i) (d) (3 + i)(2 – 5i)

1. Express each of the following in the form *a* + *ib*

(a)  (b) 

(c)  (d) 

1. Solve the following equations:
2. *x*2 + 25 = 0
3. 2*x*2 + 32 = 0
4. 4*x*2 + 9 = 0
5. *x*2 + 2*x* + 5 = 0
6. If 3 – 2i and 1 + i are two of the roots of the equation *ax*4 + *bx*3 + *cx*2 + *dx* + *c* = 0, find the values of *a*, *b*, *c*, *d* and *e.*
7. Find the square roots of the following complex numbers:

(a) 5 + 2i

(b) 15 + 8i

(c) 7 – 24i

1. Find the quadratic equations have the roots:

(a) 3i, -3i (b) 1 + 2i, 1 – 2i

(c) 2 + i, 2 – i (d) 2 + 3i, 2 – 3i

1. Find real and imaginary parts of the complex *Z* when:

(i)  = 1 + 2i

(ii) 

1. Find the modulus and principal argument of the following complex numbers

(a) 3i (b) 15 (c) -3i (d) -1

1. Find the modulus and principle argument of:

(a)  (b) 

(c)  (d) 

1. If *Z*1and *Z*2 are complex numbers, solve the simultaneous equations

4Z1 + 3Z2 = 23

Z1 + iZ2 = 6

giving your answer in the form *x* + *iy*

1. Given that 2 + *i* is a root of the equation

*Z*3 – 11*Z* + 20 = 0. Find the remaining roots.

1. Show that 1 + *i* is a root of the equation *x*4 + 3*x*2 – 6*x* + 10 = 0. Hence write down the quadratic factor of *x*4 + 3*x*2 – 6*x* + 10 and find all the roots of the equation.
2. The complex number satisfies the equation . Find the real and imaginary parts of *Z* and the modulus and argument of *Z*.
3. If *Z*1 =  and *Z*2*=* , find  and *Z*1*Z*2in the form *a* + *ib*.
4. If *Z*1=+ and *Z*2 = , find:

(i)  (ii) arg (iii) 

(iv) arg

1. One root of the equation *Z*2 + *aZ+b*=0 where *a* and *b* are real constants, is 2+3*i*. Find the values of *a* and *b.*
2. If *Z1* and *Z*2 are two complex numbers such that |*Z*1 – *Z*2| = *Z*1 + *Z*2|, show that the difference of their arguments is 
3. (a) Find the modulus and argument of 

(b) If  and . Find the moduli of *Z*1, *Z*2, *Z*1 + *Z*2 and *Z*1*Z*2.

1. Use Demoivre’s theorem to show that:



1. Use Demoivre’s theorem to show that:

cos 4θ = cos4θ – 6cos2θ sin2θ + sin4θ

sin4θ = 4cos3θ sinθ – 4cos θ sin3θ

1. Show that 
2. Use Demoivre’s theorem to find the value of 
3. Find the two square roots of *I* and the four values of .
4. Find the three roots of the equation (1 – *Z*)3 = *Z*3
5. If *W* is a complex cube root of unity, show that

(1 + *W* – *W*2)3 – (1 – *W* + *W*2)3 = 0

1. Use Demoivre’s theorem to find the four fourth roots of 8(-1 + *i*) in the form *a* + *ib*, giving *a* and *b* correct to 2 decimal places.
2. Use Demoivre’s theorem to show that

 = 1 – 12sin2*x* + 16sin4*x*

1. Prove that if  is real, the locus of the point representing the complex number *Z* in the Argand diagram is a straight line.
2. Prove that if  is purely imaginary, the locus of the point representing *Z* in the Argand diagram is a circle and find its radius.
3. If *Z* is a complex number and  = 2, find the equation of the curve in the Argand diagram on which the point representing it lie.
4. The complex numbers *Z* – 2 and *Z* – 2i have arguments which are

(i) equal and

(ii) differ by  and each argument lies between –π and π. In each case, find the locus of the point which represents *Z* in the Argand diagram and illustrate by a sketch.

1. Show by shading on an Argand diagram the region in which both |*Z* – 3 – i| ≥ |*Z* – 3 – 5*i*|

**Answers**

1. (a) 1, (b) -*i* (c)  (d) 
2. ; 512 – , 
3. ; 256 – 
4. (a) 1, (b) -1
5. 4cos3*θ* – 3cos *θ* – 4sin3*θ*, 
6. 16sin5*θ* – 20sin3*θ* + 5sin *θ*, 1 – 12sin2*θ* + 16sin4*θ*
7. .
8. 32cos6*θ* – 48cos4*θ* + 18cos2*θ* – 1, 32cos5*θ* – 32cos3*θ* + 6cos = 
9. (a) 

(b) 

(c) 

1. (a) 

(c) 

1. (a) , (b) 

13. (a)  (b) , 63.4°, , -26.6°, 3 + *i*,

4 + 3*i*.

15.  or 

or 

16. , 45°; , 90°; , 135°; , 180°

17. (i) 3 + 4*i* (ii) 25 (iii) -7 + 24*i*

18. (a) 5 + 7*i* (b) -13 + 5*i* (c) -4 (d) 1

19 (a) 7 + *i* (b) 32 – 7*i* (c) 1 – 7*i* (d) 11 – 13*i*

20. (a) 6 – 2*i* (b) 2 – 2*i* (c) -1 + *i* (d) 

21 (a) *x* ± 5*i* (b) *x* = ±4*i* (c) ±*i* (d) *x* = -1 ± 2*i*

22. *a* = 1, *b* = -8, *c* = 27, *d* = -38, *e* = 26

23. (a) ±(3 + 2*i*) (b) ±(4 + *i*) (c) ±(4 – 3*i*)

24. (a) *x*2 + 9 = 0 (b) *x*2 – 2*x* + 5 = 0

(c) *x*2 – 4*x* + 5 = 0 (d) *x*2 – 4*x* + 13 = 0

25 (i) -1, ½ (ii) 

26. (a) 3, π/2 (b) 15, 0 (c) 3, -π/2 (d) 1, *π*

27. (a) 1, -π/2 (b) , (c) , 1.25

28. 2 + 3*i* 19. 2 – *i*, -4

31. (i) Re(*Z*) = -3, Im(*Z*) = -1 (ii) , -2.82 rads

32. ; 

33. (i) 1/3 (ii)  (iii) 3 (iv) 

34. -4, 13

36. (a) 5, 0.6435 rad (b) 5, 6.5, 2.061, 32.5

41. , 

42. . 44. 

47. centre ¼ + *i*, radius 

48. 

49. (i) *x* + *y* = 2 (ii) (*x* – 1)2 + (*y* – 1)2 = 2

**Exercise 2**

Show on the Argand diagram the region represented by the following:

1. arg *z* = π,
2. arg(*z* – *i*) = π
3. arg(*z* + 1 – 3*i*) = π
4. arg(*z* – 3 + 2*i*) = π
5. arg(*z* + 2 + *i*) = π
6. arg(*z* – 1 – *i*) = *π*
7. |*z* + 1| = |*z* – 3|,
8. |*z*| = |*z* – 6*i*|
9.  = 1
10. (a) arg
11. (a) arg (b) 

In questions 12 to 24 find the Cartesian equation of the locus of the point *P* representing the complex number *z*. Sketch the locus of *P* each case.

1. 2|*z* + 1| = |*z* – 2|
2. |*z* + 4*i*| = 3|*z* – 4|
3.  = 5
4.  = 1
5.  = 5
6. 
7. *z* – 5 = λ*i*(*z* + 5), where λ is a real parameter
8.  = λ*i*, where λ is a real number.
9. *z* = 3*i* + λ(2 + 5*i*), where λ is a real parameter.
10. Im(*z*2) = 2
11. Re (*z*2) = 1
12.  = 0
13.  = 0

In questions 27 to 34 shade in separate Argand diagrams the regions represented by:

1. |*z* – *i* | ≤ 3
2. |*z* – 4 + 3*i*| < 4
3. 0 ≤ arg *z* ≤ 
4. *π* < arg *z* < *π*
5. 
6. 
7. |*z*| > |*z* + 2|
8. |*z* + *i*| ≤ |*z* – 3*i*|
9. Represent each of the following loci in an Argand diagram.

(a) arg(*z* – 1) = arg(*z* + 1)

(b) arg *z* = arg(*z* – 1 I *i*)

(c) arg(*z* – 2) = π + arg *z*

(d) arg(*z* – 1) = π + arg(*z* – *i*)

1. Find the least value of |*z* + 4| for which

(a) Re(*z*) = 5 (b) Im(*z*) = 3

(c) |*z*| = 1 (d) arg *z* = π

1. Given that the complex number *z* varies such that |*z* – 7| = 3, find the greatest and least values of |*z* – *i*|.
2. Given that the complex number *w* and *z* vary subject to the conditions |*z* – 12| = 7 and |*z* – *i*| = 4, find the greatest and least values of |*w* – *z*|.
3. In an Argand diagram, the point *P* represents the complex number *z*, where *z* = *x* + *iy*. Given that

*z* + 2 = λi(*z* + 8), where λ is a real parameter, find the Cartesian equation of the locus of *P* as λ varies. If also *z* = μ(4 + 3*i*), where λ is real, prove that there is only one possible position for *P*.

1. (i) Represent on the same Argand diagram the loci given by the equations |*z* – 3| = 3 and |*z*| = |*z* – 2|. Obtain the complex numbers corresponding to the point of intersection of these loci. (ii) Find a complex number *z* whose argument is π/4 and which satisfies the equation |*z* + 2 + *i*| = |*z* – 4 + *i*|.

**Answers**

12. *x*2 + *y*2 + 4*x* = 0, 13. *x*2 + *y*2 – 9*x* – 9*x* – *y* + 16

15. 5*x* + 3*y* = 14. 16. 2*x*2 + 2*y*2 + 25*x* + 75 = 0

17. 5*x*2 + 5*y*2 – 26*x* + 8*y* + 1 = 0.

18. *x*2 + *y*2 = 25, e*x*cluding (-5, 0)

19. *x*2 + *y*2 – 2*x* + 2*y* = 0, e*x*cluding (2, 0).

20. 5*x* – 2*y* + 6 = 0 21. *xy* = 1. 22. *x*2 – *y*2 = 1

23. *x*(*x*2 + *y*2 – 1) = 0, e*x*cluding (0, 0)

24. *y*(*x*2 + *y*2 – 9) = 0, e*x*cluding (0, 0).

34. (a) 9, (b) 3, (c) 3, (d) 4.

35. , . 38. 24, 2.

37. *x*2 + *y*2 + 10*x* + 16 = 0

38.(i)  (ii) 1 + *i*.

## Revision Exercise 3

Show on the Argand diagram the region represented by the following:

1. 
2.  = 1
3. Express the complex number  in the form *x* + *iy* where *x* and *y* are real. Given that *z*2 = 2 – 5*i*, find the distance between the points in the Argand diagram which represent *z*1 and *z*2. Determine the real numbers *α* and *β* such that *αz*1 + *βz*2 = -4 + *i*.
4. (i) Find two complex numbers *z* satisfying the equation z2 = -8 – 6*i.*

(ii) Solve the equation *z*2 – (3 – *i*)*z* + 4 = 0 and represent the solutions on an Argand diagram by vectors  and , where **O** is the origin. Show that triangle *OAB* is right-angled.

1. If *z* and *w* are complex numbers, show that:



Interpret your results geometrically.

1. A regular octagon is inscribed in the circle |*z*| = 1 in the complex plane and one of its vertices represents the number. Find the numbers represented by the other vertices.
2. (i) Two complex numbers *z*1 and *z*2 each have arguments between 0 and *π*. If *z*1*z*2 = *i* –  and  = 2*i*, find the values of *z*1 and *z*2 giving the modulus and argument of each.

(ii) Obtain in the form *a* + *ib* the solutions of the equation *z*2 – 2*z* + 5 = 0, and represent the solutions on an Argand diagram by the points *A* and *B*.

The equation *z*2 – 2*pz* + *q* = 0 is such that *p* and *q* are real, and its solutions in the Argand diagram are represented by the points *C* and *D*. Find in the simplest form the algebraic relation satisfied by *p* and *q* in each of the following cases:

1. *p*2 < *q*, *p* ≠ 1 and *A*, *B*, *C*, *D* are the vertices of a triangle;
2. *p*2 > q and 
3. (a) If –*π* < arg *z*1 + arg *z*2 ≤ *π*, show that arg(*z*1*z*2) = arg *z*1 + arg *z*2. The complex numbers  and  are represented in the Argand diagram by points *A* and *B* respectively. *O* is the origin. Show that triangle *OAB* is equilateral and find the complex number *c* which the point *C* represents where *OABC* is a rhombus. Calculate |*c*| and arg *c*.

(b) *z* is a complex number such that  where *p* and *q* are real. If arg *z* = π/2 and |*z*| = 7 find the values of *p* and *q*.

1. .
2. (a) Show that (1 + 3*i*)3 = -(26 + 18*i*).

(b) Find the three roots *z*1, *z*2, *z*3 of the equation *z*3 =-1

(c) Find in the form *a* + *ib*, the three roots *z'*1, *z'*2, *z'*3 of the equation *z*3 = 26 + 18*i*.

(d) Indicate in the same Argand diagram the points represented by *zr* and *z*'*r* for *r* = 1, 2, 3, and prove that the roots of the equations may be paired so that |*z*1 – *z*2| = *z*2 – *z*'2 = |*z*3 – *z*'3 | = 3.

1. Write down or obtain the non-real cube roots of unity, *w*1 and *w*2, in the form *a* + *ib*, where *a* and *b* are real. A regular hexagon is drawn in an Argand diagram such that two adjacent vertices represent *w*1 and *w*2, respectively and centre of the circumscribing circle of the hexagon is the point (1, 0). Determine in the form *a* + *ib*, the complex numbers represented by the other four vertices of the hexagon and find the product of these four complex numbers.
2. A complex number *w* is such that *w*3 = 1 and *w* ≠ 1. Show that:

(i) *w*2 + *w* + 1 = 0

(ii) (*x* + *a* + *b*)(*x* + *wa* + *w*2*b*)(*x* + *w*2*a* + *wb*)

is real for real *x*, *a* and *b*, and simplify this product. Hence or otherwise find the three roots of the equation *x*3 – 6*x +* 6 = 0, giving your answers in terms of *w* and cube roots of integers.

1. (i) Find, without the use of tables, the two square roots of 5 – 12*i* in the form *x* + *iy*, where *x* and *y* are real.

(ii) Represent on an Argand diagram the loci |*z* – 2| = 2 and |*z* – 4| = 7. Calculate the complex numbers corresponding to the points of intersection of these loci.

1. (i) Given that (1 + 5*i*)*p* – 2*q* = 7*i*, find *p* and *q* when (a) *p* and *q* are real (b) *p* and *q* are conjugate complex numbers.

(ii) Shade on the Argand diagram the region for which 3π/4 < arg *z* < π and 0 < |*z*| < 1. Choose a point in the region and label it *A*. If *A* represents the complex number *z*, label clearly the points *B*, *C*, *D* and *E* which represent *–z, iz,* z *+* 1 and *z*2 respectively.

1. (i) Show that *z* = 1 + *i* is a root of the equation *z*4 + 3*z*2 – 6*z* + 10 = 0. Find the other roots of the equation.

(ii) Sketch the curve in the Argand diagram defined by |*z* – 1| = 1, Im *z* ≥ 0. Find the value of *z* at the point *P* in which this curve is cut by the line |*z* - 1| = |*z* – 2|. Find also the value of arg *z* and arg(*z* – 2) at *P*.

1. (i) If *z* = 1 + *i*, find |*z*| and |*z*5|, and also the values of arg *z* and arg(*z*5) lying between –π and π. Show that Re(*z*5) = 16 and find the value of Im(*z*5).

(ii) Draw the line |*z*| = |*z* – 4| and the half line arg(*z* – *i*) = π/4 in the Argand diagram. Hence find the complex number that satisfies both equations.

1. (i) Without using tables, simplify .

(ii) Express *z*1 =  in the form *p + qi*, where *p* and *q* are real. Sketch in an Argand diagram the locus of the points representing complex numbers *z* such that |*z* – *z*1| = . Find the greatest value of *z* subject to this condition.

1. (i) Given that *z* = 1 – *i*, find the values of *r*(>0) and θ, -π < θ < π, such that *z* = *r*(cos θ + *i* sin θ). Hence or otherwise find 1/*z* and *z*6, expressing your answers in the form *p* +  *iq*, where *q*, *r* ϵ ℝ.

(ii) Sketch on an Argand diagram the set of points corresponding to the set A, where *A* = {*z*:*z* ϵ ℂ, arg (*z* – *i*) = *π*/4}. Show that the set of points corresponding to the set B, where B = {*z*:*z* ϵ ℂ, |*z* + 7*i*| = 2|*z* – 1|}, forms a circle in the Argand diagram. If the centre of this circle represents the numbers *z*1, show that *z*1 ϵ *A*.

1. Use De Moivre’s theorem to show that

cos 7*θ* = 64cos7*θ* – 112cos5*θ* + 56cos3*θ* – 7cos*θ*

1. (i) If (1 + 3*i*)*z*1 = 5(1 + *i*), express *z*1 and *z*12 in the form *x + iy*, where *x* and *y* are real. Sketch in an Argand diagram the circle |*z* – *z*1| = |*z*1| giving the coordinates of its centre.

(ii) If *z* = cos *θ* + *i* sin*θ*, show that:

Hence or otherwise, show that

16sin5*θ* = sin 5*θ* – 5sin 3*θ* + 10sin *θ*

1. .
2. (i) Given that *x* and *y* are real, find the values of *x* and *y* which make satisfy the equation 

(ii) Given that *z = x + iy*, where *x* and *y* are real, (a) Show that, the point (*x*, *y*) lies on a straight line (b) Show that, when , the point (*x*, *y*) lies on a circle with centre (-1, -½) and radius 

1. (i) Find |*z*| and arg *z* for which the complex numbers *z* given by (a) 12 – 5*i*, (b) , giving the argument in degrees (to the nearest degree) such that -180° < arg *z* ≤ 180°.

(ii) By expressing  − *i* in modulus-argument form, or otherwise, find the least positive integer *n* such that ( − *i*)*n* is real and positive.

(iii) The point *P* in the Argand diagram lies outside or on the circle of radius 4 with centre at (-1, -1). Write down in modulus form the condition satisfied by the complex number *z* represented by point *P*.

1. Sketch the circle *C* with Cartesian equation *x*2 + (*y* – 1)2 = 1. The point *P* representing the non-zero complex number *z* lies on *C*. Express |*z*| in terms of *𝜃*, the argument of *z*. Given that *z*' = 1/*z*, find the modulus and argument of *z*' in terms of *𝜃*. Show that, whatever the position of *P* on the circle *C*, the point *P*' representing *z*' lies on a certain line, the equation of which is to be determined.
2. (a) The sum of the infinite series 1 + *z* + *z*2 + *z*3 + … for values of *z* such that |*z*| < 1 is 1/(1 – *z*). By substituting *z* = ½(cos *θ* + *i*sin *θ*) in this result and using De Moivre’s theorem, or otherwise, prove that 

## Answers

3. 1 + 2*i*; ; -2, -1

4. (i) ±(1 – 3*i*), (ii) 2 – 2*i*, 1 + *i*

# 5. sum of squares of a parallelogram = sum of squares of sides TRIGONOMETRY

Trigonometry is a branch of mathematics that studies relationships involving lengths and angles of a triangle. It comes from two Greek words – *trigonom* (triangle) and *metron* (measure).

There is an enormous number of the uses of trigonometry and trigonometric functions. For instance, the technique of triangulation is used in astronomy to measure the distance between land marks. Although it was first applied in spheres, it had a greater application to planes. Surveyors have used trigonometry for many centuries.

Within mathematics, it is used in calculus (perhaps its greatest application), linear algebra, and statistics.

Trigonometric tables were created over 2000 years ago for computation in astronomy.

A student is expected to be familiar with the definitions of trigonometric ratios for acute angles.

If one angle is 90° and one of the other angles is known, the third can be determined because the three angles of any triangle add up to 180°. The two acute angles therefore add up to 90° (complimentary angles).

Once the angles are known, the ratios of the sides are determined regardless of the overall size of the triangle. If the length of one side is known, the other two are determined. These ratios are given by the following trigonometric functions of known angle, *A*; where *a*, *b*, and *c* refer to the lengths of the sides accompanying the figure.

Opposite

*a*

*A*

A

C

B

*c*

*b*

Adjacent

Hypotenuse

**Sine function (sin)**

This is the ratio of the opposite side of the triangle to its hypotenuse.



**Cosine function (cos)**

This is the ratio of the adjacent side to the hypotenuse



**Tangent function (tan)**

This is the ratio of the opposite to the adjacent side.



The hypotenuse is the side opposite to the 90° angle. It is the longest side of a triangle and one of the sides adjacent to A.

The term perpendicular and base are sometimes used for opposite and adjacent sides respectively.

Many people find it easy to remember what sides of the right angle are equal to sine, cosine, or tangent by memorising the mnemonic **SOH-CAH-TOA**.

The reciprocals of the functions are named cosecant (cosec), secant (sec) and cotangent (cot)







**Consider the following triangle ABC**

*y*

*𝜃*

A

B

*x*

C

*r*

 , ; and 

*y* = *r* sin *θ*; *x = r* cos *θ*

Applying the Pythagoras’ theorem to triangle ABC;

(*r*cos *θ*)2 + (*r* sin *θ*)2 = *r*2

*r*2cos2 *θ* + *r*2sin2 *θ* = *r*2

cos2 *θ* + sin2 *θ* = 1

**cos2 *θ* + sin2 *θ* = 1** …………………………. (i)

|  |
| --- |
| **cos2 *θ* + sin2 *θ* = 1** |

Dividing equation (i) by cos2*θ*



1 + tan2*θ* = sec2*θ*

**1 + tan2*θ* = sec2*θ*** …………………………….. (ii)

|  |
| --- |
| **1 + tan2*θ* = sec2*θ*** |

Dividing Eqn (i) by sin2*θ*



cot2*θ* + 1 = cosec2*θ*

**1 + cot2*θ* = cosec2*θ*** ……………..…………….. (iii)

|  |
| --- |
| **1 + cot2*θ* = cosec2*θ*** |

**Trigonometric Ratios for general angle**

4th quadrant

*y*

*𝜃*

*r*

3rd quadrant

2nd quadrant

1st quadrant

O

*x*

A(*x*, *y*)

Angles measured from the *x*-axis in the anti-clockwise sense are termed as positive angles while those measured in the clockwise sense are negative angles.

When A is in the 1st quadrant, *x* and *y* are positive. When A is in the 2nd quadrant, *x* is negative and *y* is positive. When A is in the third quadrant, *x* and *y* are all negative. When A is in the 4th quadrant, *x* is positive and *y* is negative. *r* is taken to be positive for all positions of the line *OA*.

The trigonometrical ratios for angles *xOA* of any magnitude are defined precisely in the same way as for acute angles.

Thus sin *𝜃* = , cos *𝜃* =  and tan *𝜃* =

The appropriate signs are attached to *x* and *y* according to the position of point A. hence for angles in which OA lies in the 1st quadrant; since *x* and *y* and *r* are positive, the sine, cosine, and tangent will all be positive.

For angles in which OA lies in the 2nd quadrant, since *y* and *r* are positive and *x* negative, the sine is positive. Cosine and tangent are negative.

For angles in which OA is in the 3rd quadrant, sine and cosine are both negative but tangent is positive. In the 4th quadrant, sine and tangent are negative while cosine is positive. This is illustrated below.

TANGENT

(+ve)

ALL

(+ve)

COSINE

(+ve)

SINE

(+ve)

**Trigonometric ratios of 30°, 45°, and 60°**

Consider the equilateral triangle *ABC* of side *x*

*x*

60°

*A*

60°

*B*

*x*

*x*

*N*

30°

30°

*C*

Considering triangle CAN:

*x*

60°

*A*

*N*

*C*

30°

Applying the Pythagoras’ theorem:

 = *x2*

 = *x*2

 = *x*2 – 

 = 



Using ***SOH-CAH-TOA***

****

****

****

****

Consider a right isosceles triangle with two sides of lengths *x* units.

*x*

45°

*B*

*C*

*A*

45°

*x*

Applying the Pythagoras’ theorem on *ABC*:

*x*2 + *x*2 = *AC*2

2*x*2 = *AC*2

*AC* = *x*

Applying *SOH-CAH-TOA*







**Example I**

Write down the values of the following, leaving surds in your answers (*the calculator should not be used*).

1. cos 780°
2. sin 780°
3. tan 780°
4. sin 540°
5. cos 540°
6. cos 210°
7. sin 150°
8. sin(-270°)
9. sin 225°
10. sin 405°
11. tan(-60°)

***Solution***

**(a) cos 780.**

60°

cos 780° = cos 60°



sin 780° = sin 60°

= 

tan 780° = tan 60° = 

**sin 540**°

sin 540° = sin 180° = 0°

cos 540° = cos 180° = 0°

**cos 210**°

30°

cos 210° = -cos 30° = 

**sin 150**°

30°

150°

sin 150 = +sin 30 = 

**sin -270**°

270°

90°

sin -270 = +sin 90° = 1

**sin 225**°

45°

sin 225° = -sin 45° = 

**sin 405°**

45°

sin 405° = sin 45° = 

## Trigonometric Curves

For any angle*θ*, a single value of sin *𝜃* or cos *𝜃* can be found. The same applies to tan *𝜃* unless when *𝜃* = ±90° and ±270° for which the values of tan *𝜃* are not defined. Thus sin *𝜃* and cos *𝜃* are functions which are defined for all negative values of *𝜃.*

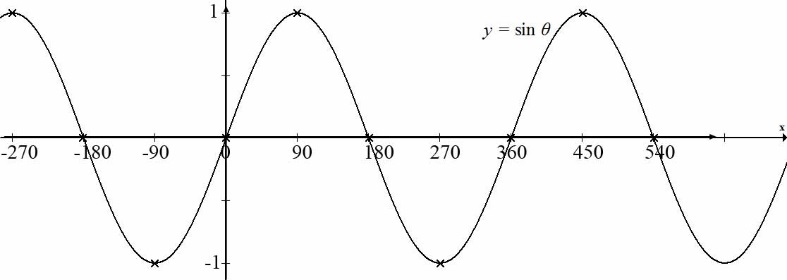
Tan θ is a function which is defined for all positive and negative values of θ except ±90° and ±270°.

To draw the graphs of sin*θ*, cos*θ* and tan*θ*, we construct a table of values, giving ordered pairs of these functions and hence plot the graph.

***Example***

*y* = sin*θ*

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *θ* | -270 | -180 | -90 | 0 | 90 | 180 | 270 | 360 | 450 | 540 |
| *y* =sin*θ* | 1 | 0 | -1 | 0 | 1 | 0 | -1 | 0 | 1 | 0 |



*y* = cos*θ*

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *θ* | -180 | -90 | 0 | 90 | 180 | 270 | 360 | 450 |
| *y* = cos *θ* | 1 | 0 | 1 | 0 | -1 | 0 | 1 | 0 |

-90

-180

0

90

270

450

*𝜃*

*y*

*y* = cos*θ*

*y* = tan*θ*

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *θ* | -270 | -180 | -90 | 0 | 90 | 180 | 270 | 360 | 450 |
| *y* =tan*θ* | ∞ | 0 | ∞ | 0 | ∞ | 0 | ∞ | 0 | ∞ |

-270

-180

-90

90

180

270

*𝜃*

*y*

*y=* tanθ

From the graph of sin *𝜃* and cos *𝜃*, the maximum values of cos *𝜃* and sin *𝜃* are 1 and 1 respectively. The minimum value of cos *𝜃* and sin *𝜃* are -1 and -1 respectively.

The graphs for sin*𝜃* and cos*𝜃* repeat themselves at regular intervals of 360° while that of tan*𝜃* repeat itself at regular interval of 180°. These intervals are called periods. These trigonometric functions are examples of periodic functions.

## Trigonometric Equations

Trigonometric equations differ from algebraic equations in that they often have unlimited number of solutions.

**Example I**

Solve the following equations for 0≤ θ ≤ 360°

1. 
2. 
3. 
4. 

***Solutions***



The acute angle whose sine is  is 30°. But sin*θ* is negative in the 3rd and 4th quadrants.

30°

30°

**(a)**

For sin θ = 

θ = 210°

θ= 330°

**(b)**  sec θ = 2





The acute angle whose cosine is  is 60° but cos θ° is positive in the 1st and 4th quadrants.

60°

30°

60°

For , *θ* = 60°, 300°

**(c)** 

The acute angle whose tangent is  is 60° but tan*θ* is negative in the 2nd and 4th quadrants.

60°

60°

 For , θ = 120°, 300°

**(d)** sin2θ = 



 and sin*𝜃* 

The acute angle whose sine is  is 45° but sin*𝜃* is positive in the 1st and 2nd quadrants.

135°

45°

45°

 For , θ = 45, 135

For 

45°

45°

For sin θ = 

*θ*° = 225°, 315°

For sin2*θ* = , θ = 45, 135°, 225°, 315°

**Example II**

Solve the following equations for -180° ≤ θ ≤ 180°.

1. Sin(2θ + 30) = 0.8
2. tan2 θ + tan θ = 0
3. sin2θ + sin θ = 0
4. 2sin2θ – sinθ – 1 = 0

***Solution***

**(a)** sin(2*θ* + 30°) = 0.8

2*θ* + 30° = sin−1(0.8)

2*θ* + 30° = 53.1°, 126.9°

 2θ = 23.1, 96.9

θ = 11.55, 48.45

For sin(2*θ* + 30°) = 0.8, θ = 11.55, 48.45.

**(b)** tan2θ + tan θ = 0

tan θ(tan θ + 1) = 0

tan θ = 0 **OR** tan θ = -1

For tan θ = 0,

θ = tan−10

θ = 0, -180, 180

For tan θ = -1, the acute angle whose tangent is 1 is 45°. But tan θ is negative in the 2nd and 4th quadrants.

45°

135°

For tan θ = -1, θ = 135°, -45°

 tan2*θ* + tan*θ* = 0

*θ* = -180°, -45°, 0, 135°, 180°

**(c) sin2** θ **+ sin** θ **= 0**

sin θ(sin θ +1) = 0

sin *θ* = 0, sin *θ* = -1

For sin θ = 0°, *θ* = 0, 180°, -180°

For sin *θ* = -1,

The acute angle whose sine is 1 is 90°. Sine is negative in the 3rd and 4th quadrants.

For sin *θ* = -1, *θ* = -90

For sin2 *θ* + sin *θ* = 0°, *θ* = -180°, -90°, 0°, 180°

**(d) 2sin2** θ **- sin θ – 1 = 0**



 sin*θ* = 1, sin*θ* = 

For sin*θ* = 1,

*θ* = sin−1(1)

*θ* = 90°

For sin*θ* = ,

*θ* = -30°, -150°

 *θ* = -30°, -150°, 90°

**Example III**

Solve the following equations from 0° to 360° inclusive.

1. cos 3*θ* = 
2. tan(3*θ* – 45°) = 
3. sec2*θ* = 3
4. 4cos2*θ* = 1
5. tan2*θ* = 
6. sin22*θ* = 1

***Solutions***

**(a) cos 3**θ = 

3θ = 

3θ = 30°, 330°, 390°, 690°, 750°, 1050°

 *θ* = 10°, 110°, 130°, 230°, 250°, 350°

**(b)** tan(3*θ* – 45°) = 

3*θ* – 45 = 

3*θ* – 45 = 26.6, 206.6, 386.6, 566.6, 746.6, 926.6

 *θ* = 23.9°, 83.9°, 143.9°, 203.9°, 263.9°, 323.9°

**(c) sec 2***θ***= 3**



2*θ* = 70.5°, 289.5°, 430.5°, 649.5°

*θ* = 35.25°, 144.75°, 215.25°, 324.75°

**(d) tan2**θ = 

tan *θ* = 

tan *θ* =  or tan *θ* = 

For tan *θ* = , *θ* = 30°, 210°

For tan θ , *θ* = 150°, 330°

 When tan2θ = , *θ* = 30°, 150°, 210°, 230°

**(e) sin22***θ* **= 1**

sin2*θ* = ±1

For sin2 *θ* = 1,

2*θ* = 90°, 450°  θ = 45°, 225°

sin 2*θ* = -1,

2*θ* = 270, 630  θ = 135°, 315°

 When sin22*θ* = 1,

*θ* = 45°, 135°, 225°, 315°

**Example IV**

Solve the following equations for values of *θ* from -180° to 180°

1. tan *θ* = cot *θ* + 3
2. sec *θ* = 2cos *θ*
3. 5sin *θ* +6cosec *θ* = 17
4. 3cos *θ* +2sec *θ* +7 = 0

***Solution***

**(a)** tan *θ* = 4cot *θ* + 3

tan *θ* =  + 3

tan2*θ*= 4 + 3tan *θ*.

tan2*θ* – 3tan *θ* – 4 = 0





tan *θ* = 4, tan *θ* = -1

When tan *θ* = 4,

*θ* = tan-1(4)

*θ* = 76°, -104° (for -180° ≤ θ ≤ 180°)

When tan *θ* = -1,

*θ* = tan-1(-1) = -45°, 135° (for -180° ≤ θ ≤ 180°)

 For tan *θ* = 4cot *θ +* 3°,

*θ* = -104°, -145°, 76°, 135°

**(b) sec***θ* = **2cos***θ*

****

1 = 2cos2*θ*

cos2*θ* =

cos *θ* = ****

For cos*θ* = , θ = 45°, -45°.

For cos*θ* = , *θ* = 135°, -135°

∴ For sec *θ* = 2cos *θ*, *θ* = -135°, -45°, 45°, 135°.

**(c) 5sin** *θ* **+ 6cosec** *θ* **= 17**

***Solution***

5sin*θ* + 6cosec*θ* = 17



5sin2*θ* + 6 = 17sin*θ*

5sin2*θ* − 17sin*θ* + 6



sin*θ* = 3

sin*θ* = 0.4

*θ* = sin-1(0.4) = 23.6, 156.4

*θ* = sin-1(3)  has no value since sin*θ* is maximum when it is 1

**(d)** 3cos *θ* +2sec *θ* +7 = 0

3cos *θ* **+**  + 7 = 0

3cos2 *θ* + 2 + 7cos *θ* = 0

3cos2 *θ* **+** 7cos *θ* + 2 = 0

****

****

cos*θ* = ****

cos*θ* = -2

For cos*θ* = -2, *θ* has no values because the minimum of cos*θ* is -1

For cos *θ* = ****

*θ* = 109.5°, -109.5°.

**Example IV**

Solve the following equations from 0° to 360°

1. 3 − cos*θ* = 2sin2*θ*
2. cos2*θ* + sin*θ* + 1 = 0
3. sec2*θ* = 3tan*θ* − 1
4. cosec2*θ* = 3 + cot *θ*
5. 3tan2*θ* + 5 = 7sec *θ*

***Solutions***

**(a)** 3 − cos*θ* = 2sin2*θ*

3 – 3cos *θ* = 2(1 – cos2*θ*)

3 – 3cos *θ* = 2 – 2cos2*θ*

2cos2*θ* – 3cos*θ* + 1 = 0

**



cos *θ* = 1, **OR** cos *θ* = 

For cos *θ* = 1,

*θ* = cos-1(1)

*θ* **=** 0°, 360°

For cos *θ* = ,

*θ* = cos-1(1/2)

*θ* = 60°, 300°

For 3 – 3cos*θ* = 2sin2*θ*, *θ* = 0°, 60°, 300°, 360°

**(b) cos2***θ* **+ sin*θ* + 1 = 0**

1 – sin2*θ* + sin *θ* + 1 = 0

sin2*θ* **-** sin *θ* – 2 = 0

sin *θ* = 



sin *θ* = 2 **OR** sin *θ* = -1

For sin *θ* = 2, the value of *θ* is not defined because sin *θ* is maximum at 1

For sin *θ*= -1, *θ* = 270°

**(c) sec2***θ* = 3tan *θ* – 1

sec2*θ* = 1 + tan2*θ*

**1 + tan2*θ* = 3tan *θ* – 1

tan2*θ* − 3tan *θ* + 2 = 0





tan*θ* = 2 **OR** tan *θ* **= 1**

For tan *θ* = 2,

*θ* = tan-1(2) = 63.4°, 243.4°

For tan *θ* = 1,

*θ* = tan-1(1) = 45°, 225°

∴ For sec2*θ* = 3tan*θ* – 1, *θ* = 45°, 63.4°, 243.4°, 225°.

**(d) cosec2***θ* = 3 + cot*θ*

But cosec2*θ* = 1 + cot2*θ*

1 + cot2*θ* = 3 + cot *θ*

cot2*θ* **– cot***θ –* 2 = 0



**

cotθ= 2 **OR** cot*θ* = -1

 **OR**   = -1

For tan*θ* = , *θ* = tan-1()

*θ* = 26.6°, 206.6°

For tan*θ* = -1, *θ* = 135°, 315°

For cosec2*θ* = 3 + cot*θ*,

*θ* = 26.6°, 135°, 206.6°, 315°

**(e) 3tan2*θ* + 5 = 7sec*θ***

3(sec2*θ* – 1) + 5 = 7sec*θ*

3sec2*θ* – 3 + 5 = 7sec*θ*

3sec2*θ* – 7sec*θ* + 2 = 0



sec*θ* = 2 **OR** sec*θ* = 

cos*θ* =  **OR** cos*θ* = 3

For cos*θ* = , *θ* = 60°, 300°

For cos*θ* = 3, *θ* is not defined because cos*θ* is maximum at 1.

**(f) 2cot2*θ* + 8 *= 7*cosec*θ***

1 + cot2*θ* = cosec2*θ*

cot2*θ* = cosec2*θ* – 1

2(cosec2*θ* – 1) + 8 = 7cosec*θ*

2cosec2*θ* – 2 + 8 = 7cosec*θ*

2cosec2*θ* – 7cosec*θ* + 6 = 0

cosec*θ* = 

cosec*θ* = 

cosec*θ*  = 3, **OR** cosec*θ* = 

sin*θ* = , **OR** sin*θ* = 2

For sin*θ* = , *θ* = 19.5, 160.5

For sin*θ* = 2, *θ* = sin-1(2)

The values of θ are not defined.

**ExampleI(UNEB Questions)**

Find all the values of *θ*, 00 ≤ *θ* ≤ 3600, which satisfy the equation

sin2 *θ* – sin 2*θ* – 3 cos2 *θ* = 0.

***Solution***

**a**) sin2 *θ* − 2 sin *θ* cos *θ* − 3cos2 *θ* = 0

Dividing through bycos2 *θ*,

tan2 *θ* − 2tan *θ* − 3 = 0

tan2 *θ* − 3tan *θ* + tan *θ* − 3 = 0

tan *θ* (tan *θ* − 3) + 1(tan *θ* − 3) = 0

(tan *θ* − 3)(tan *θ* + 1) = 0

Either tan*θ* − 3 = 0

tan *θ* = 3

*θ* = tan-1(3)

*θ* = 71.60, 251.60

Or tan*θ* + 1 = 0

tan*θ* = -1

*θ* = tan-1(-1)

*θ* = 1350, 3150

**Example II (UNEB Question)**

Solve cos *θ* + sin 2*θ* = 0 for 00 ≤ *θ* ≤ 3600.

cos *θ* + sin 2*θ* = 0

cos *θ* + 2sin *θ* cos *θ* = 0

cos *θ* (1 + 2sin *θ*) = 0

Either cos *θ* = 0

*θ* = cos-1(0)

*θ* = 900, 2700

Or 1 + 2 sin *θ* = 0

2sin *θ* = -1

sin *θ* = 

*θ* = sin-1()

*θ* = 2100, 3300

For 00 ≤ *θ* ≤ 3600, *θ* = 900, 2100, 2700, 3300

**Example III (UNEB Question)**

Solve cot2*𝜃* = 5(cosec *𝜃* + 1) for 0° ≤ *𝜃* ≤ 360°

***Solution***

**(a)** cot2 *θ* = 5(cosec θ + 1)

But cot2*θ* = cosec2*θ* – 1

cosec2 *θ* − 1 = 5(cosec *θ* + 1)

cosec2 *θ* − 1 = 5 cosec *θ* + 5

cosec2 *θ* − 5 cosec *θ* − 6 = 0

cosec2 *θ* − 6 cosec *θ* + cosec *θ* − 6 = 0

cosec *θ*(cosec *θ* − 6) + 1(cosec *θ* − 6) = 0

cosec *θ* − 6) (cosec *θ* + 1) = 0

Either cosec *θ* = 6



Or cosec *θ* + 1 = 0



Hence *θ* = 9.6°, 170.4° and 270°

**Example IV (UNEB Question)**

Solve 2sin 2*x* = 3cos *x*, for –1800  *x*  180°.

***Solution***

2 sin 2*x* = 3 cos *x*

2 sin 2*x* - 3 cos *x* = 0

But sin 2*x* = 2sin*x*cos*x*

4 sin *x* cos *x* – 3cos *x* = 0

cos *x* (4sin *x* – 3) = 0

cos *x* = 0

*x =* cos–1(0)

*x* = 900, -90

4 sin *x* – 3 = 0

sin *x* = 

*x* = 

*x* = 48.6°, 131.4°

**** *x* = (–90°, 48.6°, 90°, 131.4°) are the solutions to the equation 2sin 2*x* = 3cos *x*

**Example V (UNEB Question)**

Solve the equation cos *x* + cos 2*x* = 1 for values of *x* from 00 to 3600 inclusive

***Solution***

cos*x* + cos2*x* = 1

But cos2*x* = 2cos2 *x* – *1*

By substitution, we have

cos*x* + 2cos2*x* – 1 = 1

2cos2*x* + cos*x* – 2 = 0





Taking cos*x* =

*x* = 38.7° , 321.3°

Taking 

cos*x* = -1.280776406

(The values of *x* are not defined because *x* is maximum at 1)

Hence *x* = 38.70, 321.30

**Example VI (UNEB Question)**

Solve 7tan*θ* + cot *θ* = 5sec*θ* for 0° ≤ *θ* ≤ 180°.

***Solution***

1. 7 tan *θ* + cot *θ* = 5 sec *θ*



Multiplying through by cos*θ* sin*θ*

7sin2*θ* + cos2*θ* = 5sin*θ*

7 sin2 *θ* + 1 − sin2 *θ* = 5 sin *θ*

6 sin2 *θ* − 5 sin *θ* + 1 = 0

6sin2 *θ* − 3 sin *θ* − 2 sin *θ* + 1 = 0

3 sin *θ* (2 sin *θ* − 1) −1 (2 sin *θ* − 1) = 0

(2sin *θ* − 1)(3 sin *θ* − 1) = 0

**Either** 2 sin *θ* = 1



*θ* = 300, 1500

**Or** 3 sin *θ* − 1 = 0



*θ* = 19.50, 160.50

**19.50, 300, 1500, 160.50 are the solutions to the equation

**Example VII (UNEB Question)**

Solve the equation 4cos*x* – 2cos2*x* = 3 for 00 ≤ *x* ≤ π.

***Solution***

4 cos *x* − 2(2 cos2 *x* − 1) = 3

4 cos *x* − 4 cos2 *x* + 2 = 3

4 cos *x* − 4 cos2 *x* − 1 = 0

4 cos2 *x* − 4 cos *x* + 1 = 0

4 cos2 *x* − 2 cos *x* − 2 cos *x +* 1 = 0

2 cos *x* (2 cos *x* − 1) − 1(2 cos *x* − 1) = 0

(2 cos *x* − 1)(2 cos *x* − 1) = 0

⟹ 2 cos *x* − 1 =0

2cos *x* = 1



*x* = , .

### Elimination of *𝜃* from a set of equations

**Example**

Eliminate *𝜃* from the following equations:

1. *x* = *a* cosθ, *y* = *b* sinθ
2. *x* =*a* cotθ, *y* = *b* secθ
3. *x* = *a* tanθ, *y* = *b* tanθ
4. *x* = 1 – sinθ, *y* = 1 + cosθ
5. *x* = sinθ + tanθ, *y* = tanθ – sinθ
6. *x* cosθ + *y* sinθ = *a*, *x* sinθ – *y* cosθ = *b*

***Solution***

**(i)** *x* **=** *a* cos*θ*, *y* = *b* sin*θ*

, 

sin2*θ* + cos2*θ* = 1

** =** 1

**(ii)** *x* = *a* cot*θ*, *y* = *b* cosec*θ*

, = cosec*θ*

1 + cot2*θ* = cosec2*θ*



1 + 

**(iii)** *x =* *a* tan*θ*, *y* = *b* cos*θ*

,  

1 + tan2*θ* = sec2*θ*

1 +  = 

1 +  = **

**(iv)** *x* = 1 – sin*θ*, *y* = 1 + cos*θ*

sin*θ* = 1 – *x* , *y* – 1 =cos*θ*

sin2*θ* + cos2*θ* = 1

(1 – *x*)2 + (*y* – 1)2 = 1

** (*x* – 1)2 + (*y* – 1)2 = 1

**(v) *x* = sin***θ* **+ tan** *θ* …………………… (i)

*y* = tan*θ* – sin*θ* ………………..……(ii)

Eqn (i) + Eqn (ii);

*x* + *y* = 2tan*θ*

tan*θ* = 

Eqn (i) − Eqn (ii);

*x* – *y* = 2sin*θ*

**

From 

cot*θ* = 

From 

cosec*θ* = 

1 + cot2*θ* = cosec2*θ*

1 + 

1 + 

(*x*2 – *y*2)2 = 16*xy*

**(vi)**  *x* cos*θ* + *y* sin*θ = a …*………………..(i)

*x* sin*θ* – *y* cos*θ* = *b* ………………….. (ii)

From Eqn (i);

cos*θ* =  …………….. (iii)

Substituting Eqn (iii) in Eqn (ii);

 = *b*

*x*2sin*θ* – *ay* + *y*2sin*θ* = *xb*

(*x*2 + *y*2)sin*θ* = *xb* + *ay*

sin*θ* =  ……………………. (iv)

Substitute Eqn (iv) in Eqn (iii)

cos*θ* = 

cos*θ* = 

cos*θ* = 

cos*θ* = 

sin2*θ* + cos2*θ* = 1

 = 1

(*bx* + *ay*)2 + (*ax* – *by*)2 = (*x*2 + *y*2)2

*b*2*x*2 + 2*abxy* + *a*2*y*2 + *a*2*x*2 – 2*abxy* + *b*2*y*2 = (*x*2 + *y*2)2

(*a*2 + *b*2)*x*2 + (*a*2 + *b*2)*y*2 = (*x*2 + *y*2)2

(*x*2 + *y*2)(*a*2 + *b*2) = (*x*2 + *y*2)2

*a*2 + *b*2 = *x*2 + *y*2

**Proving Trigonometric Identities**

1. sec*θ* + cosec*θ* cot*θ* = sec*θ* cosec2*θ*
2. sin2*θ*(1 + sec2*θ*) = sec2*θ* – cos2*θ*
3. 
4. 
5. 
6. 
7. 
8. 

***Solution***

**(a) sec*θ* + cosec*θ* cot*θ***







**(b) sin2*θ*(1 + sec2*θ*)**



= sin2*θ* + tan2*θ*

= sin2*θ +* sec2*θ* – 1

= 1 – cos2*θ* + sec2*θ* – 1

= sec2*θ* – cos2*θ*

**(c)** 





**(d)** 



= 

**(e)** 







**(f)** 





**(g)** 







**(h)** 







**Formulae for sin(A ± B), cos(A ± B), and tan(A ± B)**

|  |
| --- |
| **sin(*A* + *B*) = sin*A*cos*B* + cos*A*sin*B***  **sin(*A* – *B*) = sin*A*cos*B* – cos*A*sin*B***  **cos(*A* + *B*) = cos*A*cos*B* – sin*A*sin*B***  **cos(*A* – *B*) = cos*A*cos*B* + sin*A*sin*B***  **tan(*A* + *B*) =**  **tan(*A* – *B*) =** |

**Examples**

Find the values of the following:

1. cos(45° – 30°)
2. cos 105°
3. cos 75°
4. sin(60° + 45°)
5. sin 15°

***Solution***

**(a) cos(45**° **– 35**°**)**

= cos45° cos30° + sin45°sin30°



**(b) sin(30**° **+45**°**)**

= sin30 cos45 + cos30 sin 45





**(c) cos 105**°

= cos(60° + 45°)

= cos60 cos45 – sin60 sin45





**(d) cos 75**°

= cos(30° + 45°)

= cos30° cos45° – sin30° sin45°





**(f) sin(60**° **+ 45**°**)**

= sin60 cos45 + cos60 sin45



**(f) sin 15**°

= sin(45 – 30)

= sin45 cos30 – cos45 sin30



**Example II**

If sin *A* =  and sin*B* = , where *A* and *B* are acute angles, find the values of the following:

1. **sin(*A* + *B*)**
2. **cos(*A* + *B*)**
3. **cot(*A* + *B*)**

***Solution***

*x*

3

5

*A*

*p*

5

13

*B*

|  |  |
| --- | --- |
| *x*2 + 32 = 52  *x*2 + 9 = 25  *x*2 = 16  *x* = 4 | *p*2 + 52 = 132  *p*2 + 25 = 169  *p*2 = 144  *p* = 12 |

 sin *A* = ; cos *A* = ; tan *A* = 

sin *B* = ; cos *B* = ; tan *B* = 

sin(*A* + *B*) = sin*A* cos*B* + cos*A* sin*B*

= 

= 

= 

**(b) cos(*A* + *B*) =** cos*A* cos*B* – sin*A* sin*B*

= 

= 

= 

**(c) cot(A + B)** = 

tan(A + B) = 



= 

= 

= 

= 

**Example III**

If sin*A* = , cos*B* = ,where *A* is obtuse and *B* is acute, find the values of:

1. **sin(A – B)**
2. **tan(A – B)**
3. **tan(A + B)**

***Solutions***

*x*

4

5

*A*

5

13

*B*

*p*

|  |  |
| --- | --- |
| *x*2 + 42 = 52  *x*2 + 16 = 25  *x*2 = 9  *x* = 3 | *p*2 + 122 = 132  *p*2 + 144 = 169  *p*2 = 25  *p* = 5 |

*A* is obtuse

sin *A* = ; cos*A* = ; tan*A* = 

*B* isacute

 sin*B* = ; cos*B* = ; tan*B* = 

Sin(*A – B*) = sin*A* cos*B* – cos*A* sin*B*

= 

= 

= 

**(b) tan(*A* – *B*)** = 

= 

=  = 

**(c) tan(*A* + *B*) = **



= 

= 

**Example III**

If cos*A* =  and tan*B* = ; where *A* and B are reflex angles. Find the values of:

1. **sin(*A* – *B*)**
2. **tan(*A* – *B*)**
3. **cos(*A* + *B*)**

***Solutions***

3

4

5

*A*

12

5

13

*B*

*A* and *B* are reflex

; ; tan *A* = 

cos *B* = , sin *B* = ; tan *B* = 

1. **sin(*A* – *B*)** = sin *A* cos *B* – cos *A* sin *B*

= 

= 

= 

1. **tan(*A* – *B*)** = 

= 

= 

1. **cos(A + B)** = cos *A* cos *B* – sin *A* sin*B*



**Example IV**

From the following, find the values of tan *x*

1. **sin(*x* + 45°) = 2cos(*x* + 45°)**
2. **2sin(*x* – 45°) = cos(*x* + 45°)**
3. **tan(*x* – *A*) = ,** where tan *A* = 2
4. **sin(*x* + 30°) = cos(*x* + 30°)**

***Solution***

**(a) sin(*x* + 45°) = 2cos(*x* + 45°)**

sin*x* cos45°+cos*x* sin45° = 2(cos*x* cos45°–sin*x* sin45°)









3sin *x* = cos *x*



3 tan *x* = 1

tan *x* = 

**(b)**  2sin(*x* – 45°) = cos(*x* + 45°)

2(sin*x* cos45 – cos*x*sin45) = cos*x* cos45 – sin*x*sin45

**

sin *x* − cos *x* = **cos *x* − sin *x*

sin *x* + **sin *x* = cos *x* + **cos *x*

sin *x* = cos *x*

tan *x* = 1

**(c) tan(*x* – *A*) = , tan *A* = 2**





2(tan *x* – 2) = 3(1 + 2tan *x*)

2tan *x* – 4 = 3 + 6tan *x*

4tan *x* = -7

tan *x* = 

**(d) sin(*x* + 30) = cos(*x* + 30)**

sin *x* cos 30 + cos *x* sin 30 = cos *x* cos 30 – sin *x* sin 30

sin *x* + cos *x* = cos *x* – sin *x*

sin *x* + sin *x* = cos *x* – cos *x*

sin *x* () = cos *x*()



tan *x* = 

tan *x* = 

tan *x* = 

tan *x* = 2 – 

**Example V**

Solve the following equations for 0° ≤ *θ* ≤ 360°

1. **2sin *x* = cos(*x* + 60°)**
2. **cos(*x* + 45°) = cos *x***
3. **sin(*x* – 30°) = cos *x***
4. **3sin(*x* + 10°) = 4cos(*x* – 10°)**

***Solutions***

**(a)**  **2sin *x* = cos(*x* + 60°)**

2sin *x* = cos *x* cos 60° **−** sin *x* sin 60°

2sin *x* = cos *x* – sin *x*

2sin *x* + sin *x* = cos *x*

(4 + sin *x* = cos *x*

tan *x* = 

*x* = 9.9°, 189.9°

**(b) cos(*x* + 45**°**) = cos *x***

cos *x* cos 45° – sin *x* sin 45° = cos *x*

cos *x* − sin *x* = cos *x*

**

**cos *x* = sin *x*





*x* = 67.5°, 247.5°

**(c) sin(*x* + 30) =** **cos *x***

sin *x* cos 30 – cos *x* sin 30 = cos *x*

sin *x* - cos *x* = cos *x*

sin *x* = cos *x*



tan *x* = 

*x* = 49.1°, 229.1°

**(d) 2sin(*x* + 10**°**) = 4cos(*x* – 10**°**)**

2(sin *x* cos10 − cos*x* sin10)

= 4(cos*x* cos10° + sin*x* sin 10°)

2sin*x*cos10–2cos *x* sin10=4cos *x* cos10+4sin*x*sin10

2sin *x* cos 10 – 4sin *x* sin 10

= 4cos *x* cos 10 + 2cos *x* sin 10

sin *x*(2cos 10 – 4sin 10) = cos *x*(4cos 10 + 2 sin10)





*x* = 73.4°, *x* = 253.4°

**Example VI**

If tan(*x* + 45°) = 2, find the value of tan *x*

***Solution***

tan(*x +* 45°) = 2.



 = 2

tan *x* + 1 = 2(1 – tan *x*)

tan *x* + 1 = 2 – 2tan *x*

3tan *x* = 1

tan *x* = 

**Example VII**

If tan(*A + B*) =  and tan *A* = 3, find the value of tan *B*.

tan(A + B) = 



tan *A* = 3



7(3 + tan *B*) = 1 – 3tan *B*

21 + 7tan *B* = 1 – 3tan *B*

10tan *B* = -20

tan *B* = -2

**Example VIII**

Express the following as single trigonometric ratios.

1. ****cos *x* – sin *x*
2. 
3. sin *x* + cos *x*
4. 
5. ****cos 75 + sin 75
6. 

***Solutions***

**(a)** ****cos *x* – sin *x*

= cos60 cos *x* – sin 60 sin *x*

cos(60 + *x*)

****cos *x* – sin *x* = cos(60 + *x*)

**(b)** 

= 

= tan(60 + *x*)

**(c)** sin *x* + cos *x*

= cos45 sin *x* + sin45 cos *x*

= cos(45 – *x*)

**(d)** 



= sec 39°

**(e)** ****cos 75 + sin 75

cos 60° cos 75° + sin 60° sin 75°

cos 75° cos 60° + sin 75° sin 60°

cos(75° – 60°)

cos 15°

**(f)** = 

= tan(45 – 15)

= tan(30)

**Example IX**

Prove the following identities:

1. **sin(*A* + *B*) + sin(*A* – *B*) = 2sin *A* cos *B***
2. **cos(A + B) – cos(A – B) = -2sin A sin B**
3. **tan *A +* tan *B* = **
4. **tan(A + B + C) = **

Hence prove that if *A*, *B*, and *C* are angles of a triangle, then tan *A +* tan *B* + tan *C* = tan *A* tan *B* tan *C*

***Solution***

sin (*A* + *B*) + sin(*A* – *B*)

sin *A* cos *B* + cos *A* sin *B* + sin *A* cos*B* – cos*A* sin*B*

= 2sin *A* cos *B*

sin(*A + B*) + sin(*A* – *B*) = 2sin *A* cos *B*

**(ii) cos(*A* + *B*) – cos(*A* – *B*)**

cos *A* cos *B* – sin *A* sin *B* – (cos *A* cos *B* + sin *A* sin *B*)

= -2sin A sin *B*

 cos(A + B) – cos(A – B) = -2sin A sin B

**(iii) tan A + tan B**

= 

= 

= 

tan *A +* tan *B* = 

**(iv) tan(A + B + C)**

Let *B* + *C* = D

tan(*A + D*) = 

= 

= 

= 



Since *A*, *B*, and *C* are angles of a triangle, then

*A* + *B* + *C* = 180°

tan(*A* + *B* + *C*) = tan 180°

tan(*A* + *B* + *C*) = 0



tan *A* + tan B + tan *C* – tan *A* tan *B* tan *C* = 0

tan *A* + tan B + tan *C* = tan *A* tan *B* tan *C*.

**Example (UNEB Question)**

Without using tables or calculator, evaluate tan 15°

***Solution***

tan 15° = tan(45° – 30°)



= 



= 

= 

=

**Example (UNEB Question)**

The acute angles A and B are such that cos***A*** = ½, sin ***B*** = 1/3. Show without the use of tables or calculator, show that



***Solution***

A

B

**C**

2

1

A

A

B

C

3

1

B



From compound angle formula,





****

****

### Double angle & Triple angle formulae

By writing *A* = *B* in the additional formulae for sine, cosine, and tangent, we obtain the double angle formula for each of them.

sin(A + B) = sin *A* cos *B* + cos *A* cos *B*

 sin 2*A* = sin(*A + A*)

= sin *A* cos *A* + cos *A* sin *A*

= 2sin *A* cos *A*

|  |
| --- |
| **sin 2*A* = 2sin *A* cos *A*** |

cos(*A* + *B*) = cos *A* cos *B* – sin *A* sin *B*

cos(*A + A*) = cos *A* cos *A* – sin *A* sin *A*.

= cos2*A* – sin2*A*

But cos2*A* = 1 – sin2*A*

cos 2*A* = 1 – sin2*A* – sin2*A*

= 1 – 2sin2*A*

But when sin2*A* = 1 – cos2*A*

cos2*A* = cos2A – sin2*A*

= cos2*A* – (1 – cos2*A*)

= 2cos2*A* – 1

|  |
| --- |
| **cos 2*A* = 2cos2A – 1 OR**  **cos 2*A* = 1 – 2sin2A** |

tan(*A* + *B*) = ; where *A = B*

tan(A + A) = 

= 



**sin 3*A* = sin(*A* + 2*A*)**

= sin *A* cos 2*A* + cos *A* sin 2*A*

= sin *A(*1 – 2sin2*A*) + cos A(2sin *A* cos *A*)

= sin *A* – 2sin3*A* + 2cos2A sin*A*

= sin *A* – 2sin3*A* + 2(1 – sin2*A*)sin *A*

= sin *A* – 2sin3*A* + 2sin *A* – 2sin3*A*

= 3sin *A* – 4sin3 *A*

|  |
| --- |
| **sin 3*A* = 3sin *A* – 4sin3*A*** |

**cos 3*A* = cos(2*A* + *A*)**

= cos2*A* cos *A* – sin 2*A* sin *A*

= (2cos2*A* – 1)cos *A* – (2sin *A* cos *A*)sin *A*

= 2cos3*A* – cos *A* – 2sin2*A* cos*A*

= 2cos3*A* – cos *A* – 2(1 – cos2*A*)cos*A*

= 2cos3*A* – cos *A* – 2cos *A* + 2cos3*A*

= 4cos3*A* – 3cos *A*

** cos 3*A* = 4cos3*A* – 3cos *A***

**tan 3*A* = tan(*A +* 2*A*)**

**= **

**= **

**= **

**= **

**= **

|  |
| --- |
|  |

**Example I**

Simplify the following expressions

1. **2sin 17 cos 17**
2. ****
3. **2cos242 – 1**
4. 
5. 1 – 2sin2
6. ****
7. **1 – 2sin23***θ*
8. ****
9. **sec** *θ* **cosec** *θ*
10. **2sin 2*A* cos 2*A***

***Solutions***

**(i) sin 2(17°)** = 2sin 17° cos 17°

sin 34° = 2sin 17° cos 17°

 2sin 17 cos 17 = sin 34

**(ii) tan(30**° **+ 30**°**) = **

tan 60° = 

 = tan 60°

**(iii) 2cos242**° **– 1**

cos 2*θ* = 2cos2*θ* – 1

cos 2(42°) = 2cos242° – 1

cos 84° = 2cos242° – 1

2cos242° – 1 = cos 84°

**(iv) 2sin*𝛉* cos*𝛉***

sin 2*θ* = 2sin *θ* cos *θ*

sin 2 = 2sin cos

sin θ = 2sin cos

sin *θ* = 2sincos

2sincos = sin *θ*

**(v) 1 – 2sin222½°**

cos 2*A* = 1 – 2sin2*A*

cos 2(22½) = 1 – 2sin222½

cos 45 = 1 – 2sin2 22½

1 – 2sin2 22½ = cos 45

**(vi) **

**=** tan

= tan *θ*

**(vii)** 1 – 2sin2*θ*

cos2(3*θ*) = 1 – 2sin2 3*θ*

cos 6*θ* = 1 – 2sin23*θ*

1 – 2sin23*θ* = cos 6*θ*

**(viii) **

tan 40 = tan(20 + 20)









**(ix) **



But sin 2θ = 2sin*θ* cos*θ*

sin 2θ = sin θ cos θ



**(x) 2sin 2*A* cos 2*A***

sin 4*A* = sin2(2*A*)

= 2sin 2*A* cos2*A*

2sin 2*A* cos 2*A* = sin 4*A*

**Example II**

Evaluate the following without using tables or calculator:

1. 2sin 15° cos 15°
2. 2cos275° – 1
3. cos2 22½° - sin2 22 ½°
4. 
5. 
6. 

***Solution***

**(a) 2sin 15**° **cos 15**° **= sin 2(15**°**)**

= sin 30°

= 

**(b) 2cos275° – 1 = cos 150°**

**= -cos 30°**

**= **

**(c) cos2 22½° - sin2 22 ½°**

= cos(22° + 22

= cos 45

= 

**(d)** 



**(e)** 



**(f)** = cos 135°

= -cos 45°

= 

**Example III**

Solve the following equations from 0 ≤ *θ* ≤ 360°

1. **cos 2θ + cos *θ* + 1 = 0**
2. **sin 2*θ* cos *𝛉* + sin2*𝛉* = 1**
3. **2sin θ(5cos 2*θ* + 1) = 3 sin 2*θ***
4. **3cot 2**θ + cot θ = 1
5. **4tan** *θ* **tan 2*θ* = 1**

***Solution***

**(a) cos 2θ + cos *θ* + 1 = 0**

****

For = 120°, 240°

The solutions to the equation

cos 2θ + cos *θ* + 1 = 0 are 90°, 120°, 240° and 270°.

**(b) sin 2*θ* cos *𝛉* + sin2*𝛉* = 1**





For , *θ* = 90°

For , *θ* = 270°

For , θ = 30°, 150°

30°, 90°, 150°, 270° are the solutions to the equation sin 2*θ* cos *𝛉* + sin2*𝛉* = 1

**(c) 2sin *θ*(5cos 2*θ* + 1) = 3 sin 2*θ***





For sin *𝜃* = 0, *𝜃* = 0°, 180°, 360°

For cos *𝜃* = , *𝜃* = 120°, 240°

For cos *𝜃* = 0.8, *𝜃* = 36.9°, 323.1°

0, 36.9, 120, 180, 240, 323.1, 360 are the solutions to the equation

2sin *θ*(5cos 2*θ* + 1) = 3 sin 2*θ*

**(d) 3cot 2*θ* + cot *θ* = 1**

****

****

****

For tan*θ* = 1, *θ* = 45°, 225°

For tan*θ* = , θ = 121°, 301°.

**(e)** **4tan** *θ* **tan 2*θ* = 1**







When , θ = 18.4°, 198.4°

When , *θ* = 161.6°, 341.6°

### *t*-formula

|  |
| --- |
| **If *t* = ,**  **,**  **And if *t* = tan *x***  **,** |

***Proof***

If *t* = ,





Dividing through by 





|  |
| --- |
|  |





Dividing through by 





For *t* = tan *x*



Dividing through by 









**Note:** The *t*-formula is used to solve equations of the form 

**Example I**

Solve the following equations for 0 ≤ θ ≤ 360°

1. ****
2. ****
3. ****
4. ****

***Solution***

**(a) **

, , for 



2(1 – *t*2) + 3(2*t*) = 2(1 + *t*2)

2 – 2t2 + 6*t* = 2 + 2*t*2

4*t*2 – 6*t* = 0

2*t*(2*t* – 3) = 0

*t* = 0

= 0 and 

For , 

= 0°, 180°, …

*θ* = 0, 360.

For , = 56.3°

*θ* = 112.6°

0°, 112.6°, and 360° are solutions to the equation ****

**(b) **



3 – 3*t*2 – 8*t* + 1 + *t*2 = 0

-2*t*2 – 8*t* + 4 = 0

*t*2 + 4*t* – 2 = 0







For *t* = , 

= 102.7, 282.7

*θ* = 205.4°

When *t* = 

= tan-1(-2 + )

= 24.2°

*θ* = 48.4°

*θ* = 48.4° and 205.4° are the solutions to the equation

**(c) **



3 – 3*t*2 + 8*t* = 2(1 + *t*2)

3 – 3*t*2 + 8*t* = 2 + 2*t*2

5*t*2 – 8*t* – 1 = 0



*t* = -0.11652

*t* = 1.71652

For *t* = -0.11652, = -0.11652

= 173.4 *θ* = 346.7°

tan= 1.71652

= 59.8° *θ* = 119.6°

119.6° and 346.7° are solutions to the above equation.

**(d)** 4cos *θ* sin *θ* + 15 cos2*θ* = 10

2 × 2sin*θ* cos *θ* + 15cos 2*θ* = 10

2sin 2*θ* + 15cos 2*θ* = 10

2sin 2*θ* + 15cos 2*θ* = 0

Let *t* = tan *θ*

 and 



4*t* + 15 – 15*t*2 = 10 + 10*t*2

25*t*2 – 4*t* – 5 = 0



*t* = 0.5343

*t =* -0.3743

For *t* = 0.5343

tan *θ* = 0.5343

θ = 28.1°

θ = 208.1°

For *t* = -0.3743, tan *θ* = -0.3743

*θ* = tan-1(0.3743)

*θ* = 159.5°, 200.5°

28.1°, 208.1°, 159.5° and 200.5° are the solutions to the above equation

## The *R*- Formula

The *R*-formula is used to solve equations of the form 

|  |
| --- |
|  |

Where *R* =  and 

**Example I**

Solve the equation 3cos*θ* + 4sin*θ* = 2 for 0 ≤ θ ≤ 360°

***Solution***





By comparison



 ……………………….. (i)

 ……………………….. (ii)

Eqn (ii) ÷ Eqn (1);











*θ* – 53.1° = 66.4°, 293.6°

*θ* – 119.5°, 346.7°

***Alternatively***

3cos *θ* + 4sin *θ* = 2

*R*cos(*θ* – α) = 2





*α* = 53.1

5cos(θ – 53.1) = 2

cos(*θ* – 53.1°) = 

*θ* – 53.1° = 66.4°, 293.6°

*θ* = 119.5°, 346.7°

**Example II**

sin *θ* + cos*θ* = 1 for 0 ≤ *θ* ≤ 360

***Solution***







*R* sin(*θ* + *α*) = 1

2sin(*θ* + 60°) = 1

sin(*θ* + 60°) = 

*θ* + 60° = sin(½)

*θ* + 60° = 30, 150°

*θ* = -30, 90°

 *θ* = 90°, and 330°.

**Example III**

cos *θ* – 7sin *θ* = 2 for 0° ≤ θ ≤ 360°

***Solution***

cos *θ* – 7sin *θ* = 2









*θ* + 81. 9° = 73.6°, 286.4°

*θ* = -8.3°, 204.5°

**** *θ* = 204.5°, 351.7°

**Example IV**

Solve: 5sin*θ* – 12cos*θ* = 6

***Solution***

*R*sin(*θ* – *α*) = 6





13sin(*θ* – 67.4) = 6

sin(*θ* – 67.4) = 

*θ* – 67.4° = 27.5°, 152.5°

*θ* = 94.9°, 219.9°

**Example V**

Solve cos*θ* + sin*θ* = sec*θ* for 0 ≤ *θ* ≤ 360°

***Solution***



But cos 2*θ* = 2cos2*θ* – 1



Subsituting for cos2θ and sin θ cos θ in Eqn (i);

sin 2*θ* = 2sin*θ* cos*θ*

sin*θ* cos *θ* =sin2*θ*



cos 2*θ* + sin 2*θ* = 1

*R*cos(2*θ* – α) = 1







*θ* = 45°, 180°, 225°

**Example VI**

Solve the equation 4cos*θ* sin*θ* + 15cos 2*θ* = 10

***Solution***

4cos*θ* sin*θ* + 15cos 2*θ* = 10

2(2sin*θ* cos*θ*) + 15cos2*θ* = 10

2sin2*θ* +15cos2*θ* = 10

*R* sin(2*θ* + *α*) = 10

**

**sin(2*θ* + *α*) = 10



**sin(2*θ* + 82.4°) = 10

sin(2*θ* + 82.4°) = 

2*θ* + 82.4° = 

2*θ* + 82.4° = 41.4°, 138.6°, 401.4°, 498.4°

*θ* = 339.5°, 28.1°, 159.5°, 208°

**Example VII**

Show that 3cos*θ* + 2sin*θ* can be written as cos(θ – α). Hence find the minimum and maximum values of the function, giving the corresponding values of θ from -180° to 180°

***Solution***

3cos*θ* + 2sin*θ*

*R*cos(*θ* – *α*)







3cosθ + 2sinθ = *R* cos(θ – α)



Let *y* 

For the maximum value of *y*, 



And for minimum value of *y*, = -1



For *y*max cos(*θ* – 33.7°) = 1,

θ – 33.7° = cos-1(1)

θ – 33.7° = 0, 360°.

θ = 33.7°

For *y*min  cos(*θ* – 33.7°) = -1,

θ – 33.7° = 180°.

θ = 213.7°

**Example VII**

Find the maximum and minimum values of the following expressions, stating the value of θ for which they occur (from 0° to 360°)

1. 8cos*θ* – 15sin*θ*
2. 4sin*θ* – 3cos*θ*
3. sin*θ* – 6cos*θ*
4. cos(*θ* + 60) – cos*θ*

***Solution***

**(a)** 8cos*θ* – 15sin*θ*

***R*** cos(θ – α)

******

****

17cos(*θ* – 61.9°)

Let *y* = 17cos(*θ* – 61.9°)

For *y*max, cos(*θ* – 61.9°) = 1

*y*max = 17

*θ* – 61.9° = cos-1(1)

*θ* – 61.9° = 0, 360°

*θ* = 61.9°

For *y*min, cos(*θ* – 61.9) = -1

*y*min = -17

*θ* – 61.9° = cos-1(-1)

*θ* – 61.9° = 180°

*θ* = 241.9°

**(b)** 4sin*θ* – 3cos*θ*

****

*R* sin(*θ* – *α*)

5 sin(*θ* – *α*)



5 sin(*θ* – 36.9°)

Let *y* = 5 sin(*θ* – 36.9°)

*y*min = -5

*y*max = 5

For *y*min, sin(*θ* – 36.9°) = -1

*θ* – 36.9° = 270°

*θ* = 306.9°

For *y*max, sin(*θ* – 36.9°) = 1

*θ* – 36.9° = 90°

*θ* = 126.9°

**(c) sin*θ* – 6cos*θ***



sin(*θ* – *α*)



*y* = sin(*θ* – 80.1)

*y*max =  and it occurs when sin(*θ* – 80.1) = 1

*θ* – 80.1° = 90°

*θ* = 170.5°

*y*min =  and it occurs when

sin(*θ* – 80.1) = -1

*θ* – 80.1° = 270°

*θ* = 350.5°

**(d) cos(*θ* + 60) – cos*θ***

= cos*θ* cos 60 – sin*θ* sin60 – cos*θ*

= cos*θ* - sin*θ* − cos*θ*

= 

*y* =

*y* = 

**

*y* = 

**

*y* = -[cos(*θ* – 60)]

*y*min occurs when cos(*θ* – 60) = 1

*θ* – 60° = 0, 360

*θ* = 60°

*y*max = 1 and occurs when cos(*θ* - 60°) = -1

*θ* – 60° = cos-1(-1)

*θ* = 240°

**Example VIII (UNEB Question)**

Solve  for 0 ≤ θ ≤ *π*

***Solution***



*R*cos(*θ* – *α*) = 2



2cos(*θ* – *α*) = 2



2 cos(*θ* – 60°) = 2

cos(*θ* – 60°) = 1

*θ* – 60° = cos-1(1)

*θ* – 60° = 0

*θ* = 60°



Since 180 = π radians, 

**Example IX (UNEB Question)**

1. Express 4cos*θ* – 5sin*θ* in the form R cos (*θ* + β), where *R* is a constant and β an acute angle.

Determine the maximum value of the expression and the value of *θ* for which it occurs

1. Solve the equation 4 cos *θ* – 5 sin *θ* = 2.2,

for 00 < *θ* <3600.

***Solution***

4cos*θ* – 5sin*θ*

*R*cos(θ + β)

*β* = 



cos(*θ* + 51.3°)

Let *y* = cos(*θ* + 51.3°)

*y*max = and it occurs when cos(*θ* + 51.3°) = 1

*θ* + 51.3° = 0

*θ* = -51.3°

 *θ* = 308.7° (00 < *θ* <3600)

4cos*θ* – 5sin*θ* = 2.2

 cos(*θ* + 51.3°) = 2.2

cos(*θ* + 51.3°) = 

*θ* + 51.3° = 69.9°, 290.1°

*θ* = 18.6°, 238.8°

**Example XI (UNEB Question)**

Express *y* = 8cos*x* + 6sin *x* in the form R cos (*x –* ) where *R* is positive and is acute . Hence find the maximum and minimum values of 

***Solution***

8cos*x* + 6sin *x* = *R*cos(*x* – *α*)

8cos*x* + 6sin *x* = *R* cos *x* cos *α* + *R* sin *x* sin α

By comparison

*R*cos *α* = 8 …............………………… (i)

*R*sin *α* = 6……………............……….(ii)

Eqn (i)2 + Eqn (ii)2;

*R*2 = 82 + 62 = 100

*R* = 10

Eqn (ii)  Eqn (i)



Hence 8cos*x +* 6sin*x =* 10cos(*x* − 36.870)



***Note***: For *y* to be maximum, the denominator must be minimum and for *y* to be minimum, the denominator must be maximum.

Let 





The maximum and minimum values of  are 0.2 and 0.04 respectively.

## Factor Formula

|  |
| --- |
| 1. **sin P + sin Q =** 2. **sin P − sin Q =** 3. **cos *P +* cos Q =** 4. **cos *P –* cos Q = -** |

**Application of the factor formula**

Example 1

Express the following in factors:

1. sin 7*θ* + sin 5*θ*
2. sin 4*x* – sin 2*x*
3. cos 7*x* + cos 5*x*
4. cos 3A – cos 5*A*
5. sin(*x* + 30) + sin(*x* – 30)
6. cos(*x* + 30) – cos(*x* – 30)
7. cos *x* – cos 
8.  + cos 2*θ*
9. 1 + sin 2*x*
10. Sin 2(*x* + 40) + sin 2(*x* – 40)

***Solution***

**(a)** sin 7*θ* + sin 5*θ*

From sin *P* + sin Q = 2sin ()cos ()

sin 7*θ* + sin 5*θ* = 2sin ()cos 

= 2 sin 6*θ* cos *θ*

**(b)** sin 4*x* – sin 2*x*

From sin P – sin Q = 2cos  sin 

sin 4*x* – sin 2*x* = 2cossin

sin 4*x* – sin 2*x* = 2cos 3*x* sin *x*

**(c)** cos 7*x* + cos 5*x*

From cos *P* + cos *Q* = 2cos cos

Cos 7*x* + cos 5*x* = 2coscos

= 2cos 6*x* cos *x*

**(d)** cos 3*A* – cos 5*A*

From cos *P* – cos *Q* = -2sinsin

cos 3*A –* cos 5*A* = -2 sin  sin 

= -2 sin 4*A* sin (−*A*)

= 2 sin 4*A* sin *A*

**(e)** sin(*x* + 30) + sin(*x* – 30)

= -2sincos

= -2 sin *x* cos 30

**(f)** cos(*x* + 30) – cos(*x* – 30)

= 2sinsin

= 2sin*x* sin30

**(g)** cos - cos  = −2sin sin 

= 2sin *x* sin 

**(h)**  + cos 2*θ*

cos 60 + cos 2*θ*

= 2coscos

= 2cos(30 + *θ*) cos(30 – *θ*)

**(i)**  1 + sin 2*x*

sin 90 + sin 2*x*

2sincos

= 2sin(45 + *x*)cos(45 – *x*)

**(j)**  sin2(*x* + 40) + sin2(*x* – 40)

= 2sincos

= 2sin 2x cos 80

**Example II**

Solve the following equations from *x* = 0° to 360° inclusive.

1. cos *x* + cos 5*x* = 0
2. sin 3*x* – sin *x* = 0
3. sin (*x* + 10) + sin *x* = 0
4. cos(2*x* + 10) + cos(2*x* – 10) = 0
5. cos(*x* + 20) – cos(*x* – 70) = 0

***Solution***

**(a)** cos *x* + cos 5*x* = 0

2coscos = 0

2cos 3*x* cos−2*x* = 0

2cos 3*x* cos 2*x* = 0

cos 3*x* cos 2*x* = 0

 cos 2*x* = 0 **OR**

cos 3*x* = 0

For cos 2*x* = 0;

2*x* = cos−1(0)

2*x* = 90°, 270°, 450°, 630°, 810°

 *x* = 45°, 135°, 225°, 315°.

For cos 3*x* = 0;

3*x* = cos−1(0)

3*x* = 90°, 270°, 450°, 630°, 810°, 990°, 1170°

*x* = 30°, 90°, 150°, 210°, 270°, 330°.

∴ The solutions to the equation cos*x* + cos 5*x* = 0 are 30°, 45°, 90°, 135°, 150°, 210°, 225°, 270°, 315°, 330°.

**(b)** sin 3*x* – sin *x* = 0

2cossin = 0

2cos2*x* sin *x* = 0

cos2*x* sin *x* = 0

2*x* = cos−1(0)

2*x* = 90°, 270°, 450°, 630°, 810°, 990°

*x* = 45°, 135°, 225°, 315°

And for sin *x* = 0;

*x* = sin−1(0)

*x* = 0, 180°, 360°

 The solutions to the equation sin 3x – sin x = 0 are 0, 45, 135, 180, 225, 315, 360.

**(c)** sin(*x* + 10) + sin *x* = 0

2sincos = 0

2sin(*x* + 5) (cos 5) = 0

sin (*x* + 5) = 0

*x* + 5 = sin−10)

*x* + 5 = 0, 180°, 360°

*x* = 355°, 175°.

*x* = 175°, 335° are solutions to the equation

sin(*x* + 10) + sin *x* = 0

**(d)** cos(2*x* + 10) + cos(2*x* – 10) = 0



2cos 2*x* cos 10 = 0

cos 2*x* = 0

2*x* = cos−1(0)

2*x* = 90°, 270°, 450°, 630°.

*x* = 45°, 135°, 225°, 315°

The solutions to the equation cos(2*x* + 20) + cos(2*x* – 10) = 0 are *x* = 45°, 135°, 225° and 315°

**(f)** cos(*x* + 20) – cos(*x* – 70) = 0

−2sinsin = 0

−2sin(*x* – 25)sin 45 = 0

sin(*x* – 25) = 0

*x* – 25 = sin−1(0)

*x* – 25 = 0, 180°, 360°

*x* = 25, 205°, 385°

**Example II**

Prove the following identities:

(a) 

(b) 

(c) 

(d) 

***Solution***

**(a)** 





**(b)** 



**(c)** 



**(d)** 





**Example IV**

Prove the following

1. sin *x* + sin 2*x* + sin 3*x* = sin 2*x*(2cos*x* + 1)
2. cos *x* + sin 2*x* – cos 3*x* = sin 2*x*(2sin*x* + 1)
3. cos*θ* – 2cos 3*θ* + cos 5*θ* = 2sin*θ* (sin 2*θ* – sin 4*θ*)
4. sin *x* – sin(*x* + 60) + sin(*x* + 120) = 0
5. 1 + 2cos 2*θ* + cos 4*θ* = 4cos2*θ* cos 2θ

***Solutions***

**(a)**  sin *x* + sin 2*x* + sin 3*x*

= sin *x* + sin 3*x* + sin 2*x*

= 2sincos + sin 2*x*

= 2sin 2*x* cos(−*x*) + sin 2*x*

= 2sin 2*x* cos *x* + sin 2*x*

= sin 2*x*(2cos *x* + 1)

 sin *x* + sin 2*x* + sin 3*x* = sin 2*x*(2cos *x* + 1)

**(b)** cos *x* + sin 2*x* – cos 3*x*

= cos *x* – cos 3*x* + sin 2*x*

= −2sinsin + sin 2*x*

= −2sin 2*x* sin(-*x*) + sin 2*x*

= 2sin 2*x* sin *x* + sin 2*x*

= sin 2*x*[2sin *x* + 1]

 cos *x* + sin 2*x* – cos 3*x* = sin 2*x*[2sin *x* + 1]

**(c)** cos*θ* − 2cos 3*θ* + cos 5*θ*

= cos *θ* − cos3*θ* + cos 5*θ* − cos 3*θ*

= −2sin 2*θ* sin(-*θ*) + −2sin 4*θ* sin *θ*

= 2sin 2*θ* sin *θ* – 2sin 4*θ* sin*θ*

= 2sin *θ* (sin 2*θ* – sin4*θ*)

 cos*θ* − 2cos 3*θ* + cos 5*θ* = 2sin *θ* (sin 2*θ* – sin4*θ*)

**(d)** sin *x* – sin(*x* + 60) + sin(*x* + 120)

= sin *x* + sin(*x* + 120) – sin(*x* + 60)

= 2sin(*x* + 60)cos -60 – sin(*x* + 60)

= sin(*x* + 60) – sin(*x* + 60)

= 0

 sin *x* – sin(*x* + 60) + sin(*x* + 120) = 0

**(e)** 1 + 2cos 2*θ* + cos 4*θ*

Since cos 4*θ* = cos22*θ* − 1,

 1 + 2cos 2*θ* + 2cos22*θ* – 1

= 2cos 2*θ* + 2cos22*θ*

= 2cos2*θ* [1 + cos 2*θ*]

= 2cos 2*θ* [1 + 2cos2*θ* – 1]

= 4 cos2*θ* cos 2*θ*

 1 + 2cos 2*θ* + cos 4*θ* = 4 cos2*θ* cos 2*θ*

**Example V**

Solve the following equations for values of *θ* from 0° to 180° inclusive

1. cos *θ* + cos 3*θ* + cos 5*θ* = 0
2. sin *θ* − 2sin 2*θ +* sin 3*θ* = 0
3. sin *θ +* cos 2*θ* − sin 3*θ* = 0
4. sin 2*θ* + sin 4*θ* + sin 6*θ* = 0
5. cos*θ* + 2cos*θ* + cos*θ* = 0

***Solution***

1. cos *θ* + cos 3*θ* + cos 5*θ* = 0

cos *θ* + cos 5*θ* + cos 3*θ =* 0

2cos3 *θ* cos−2 *θ* + cos3*θ* = 0

cos 3 *θ(2*cos2 *θ* + 1] = 0

Either cos 3*θ* = 0 **OR**

cos 2*θ* = −

For cos 3*θ* = 0;

3*θ* = cos−1(0)

3*θ* = 90°, 270°, 450°, 630°, 810°, 990°

*θ* = 30°, 90°, 150°, 210°, 270, 330°

 *θ* = 30°, 90°, 150° (for 0° ≤ *θ* ≤ 180°)

For cos 2*θ* = −;

2*θ* = cos−1()

2*θ* = 120°, 240°.

*θ* = 60°, 120°

30°, 60°, 90°, 120°, 150° are the solutions to the equation cos 3*θ* + cos 3*θ* + cos 5*θ* = 0

**(b)** sin *θ* − 2sin 2*θ +* sin 3*θ* = 0

sin *θ* + sin 3*θ* – 2sin 2*θ* = 0

2sin 2*θ* cos(-*θ*) – 2sin 2*θ* = 0

2sin 2*θ* cos *θ* – 2sin 2*θ* = 0

2sin 2*θ* (cos *θ* – 1) = 0

**Either** sin 2*θ* = 0 **OR** cos *θ* = 1

For sin 2*θ* = 0;

2*θ* = sin−10

2*θ* = 0°, 180°, 360°

 *θ* = 0°, 90°, 180°

**(c)** sin *θ +* cos 2*θ* − sin 3*θ* = 0

sin *θ* – sin 3*θ* + cos 2*θ* = 0

2cos 2*θ* sin -*θ* + cos 2*θ* = 0

cos 2*θ*(-2sin *θ* + 1) = 0

cos 2*θ* = 0 **OR** sin*θ* = 

For cos 2*θ* = 0

2*θ* = cos-10

2*θ* = 90°, 270°, 450°

= 45°, 135°

For sin *θ* = ;

*θ* = sin-1()

*θ* = 30°, 150°

* 30°, 45°, 135°, 150° are the solutions to the equation sin *θ +* cos 2*θ* − sin 3*θ* = 0

**(d)**  sin 2*θ* + sin 4*θ* + sin 6*θ* = 0

(sin 2*θ* + sin 6*θ*) + sin 4*θ* = 0

2sin 4*θ* cos -2*θ* + sin 4*θ* = 0

2sin 4*θ* cos 2*θ* + sin 4*θ* = 0

sin 4*θ* (2cos 2*θ* + 1) = 0

For sin 4*θ* = 0;

4*θ* = sin-10

4*θ* = 0, 180, 360, 540, 720

= 0, 45, 90, 135, 180

For 2cos 2*θ* + 1 = 0

cos 2 *θ* = 

2*θ* = 120°, 240°

*θ* = 60°, 120°

* 0°, 45°, 60°, 90°, 120°, 135°, 180° are the solutions to the equation sin 2*θ* + sin 4*θ* + sin 6*θ* = 0

**(e)** cos*θ* + 2cos*θ* + cos*θ* = 0

cos*θ* + cos *θ* + 2cos *θ* = 0

2coscos(-*θ*) + 2cos  = 0

2cos(cos *θ* + 1) = 0

cos = 0

 = cos-1(0)

 = 90, 270, 450

*θ* = 60, 180

**For** (cos*θ* + 1) = 0;

cos *θ* = −1

*θ* = cos-1(−1)

*θ* = 180

 60, 180 are the solutions to the equation

cos*θ* + 2cos*θ* + cos*θ* = 0

**Example V**

Prove the following identities if A, B and C are taken to be angles of a triangle.

1. sin *A* + sin(B – C) = 2sin B cos C
2. cos A – cos (B – C) = -2cos B cos C
3. sin A + sin B + sin C = 4coscoscos
4. sin 2A + sin 2B + sin 2C = 4sin A sin B sin C
5. cos A + cos B + cos C – 1

=

***Solutions***

Sin A + sin(B – C)

= 2sincos

= 2sincos

But *A* + *B* + *C* = 180

*A* + *B* + *C* – 2*C* = 180 – 2*C*

*A* + *B* – C = 180 – 2*C*

 = 90 – *C*

sin = sin(90 – *C*)

= sin 90 cos *C* – cos 90 sin *C*

= cos *C*

*A* + *B* + C = 180

*A* + *C* + *B* – 2*B* = 180 – 2*B*

*A + C – B =* 180 – 2*B*

cos = cos

cos = cos(90 – *B*)

= cos 90 cos *B* + sin90 sin *B*

= sin *B*

 sin *A* + sin(*B* – *C*) = 2sin *B* cos *C*

**(c)** sin A + sin B + sin C

= sincos + sin C

= 2sincos + 2sincos

But A + B + C = 180

C = 180 – (A + B)

 = sin (90 - )

sin  = sin 90 cos - cos 90 sin

= cos

cos  = cos(90 - )

cos = cos 90 cos + sin 90 sin

= sin

2coscos + 2coscos

= 2coscos+2coscos

= 

= 2cos[2coscos]

= 4coscoscos.

sin A + sin B + sin C = 4coscoscos.

**(d)**  sin 2*A* + sin 2*B* + sin 2*C*.

= 2sin(*A* + *B*) cos(*A* – *B*) + 2sin *C* cos *C*

But *A* + *B* + *C* = 180

*C* = 180 – (*A* + *B*)

sin *C* = sin[180 – (*A* + *B*)]

sin *C* = sin 180 *c*os (*A* + *B*) – *c*os 180 sin(*A* + *B*)

sin *C* = sin(*A* + *B*)

cos *C* = *c*os(180 – (*A*+*B*))

cos *C* = *c*os 180 cos(*A* + *B*) + sin 180 sin(*A* + *B*)

= −cos(*A* + *B*)

2sin(A + B)cos(A – B) + 2sin C cos C

= 2sin C cos (A – B) + 2sin C(-cos(A+B))

= 2sin C[cos(A – B) – cos(A + B)]

= 2sin C[-2sin A sin –B]

= 4sinA sin B sin C

 sin 2A + sin 2B + sin 2C = 4sin A sin B sin C

**(e)** cos *A* + cos *B* + cos *C* − 1

cos *C* = 2cos2 - 1

cos *C* = 1 – 2sin2

2sin2 = 1 – cos *C*

cos *A* + cos *B* + cos *C* – 1 = cos *A* + cos *B* – 2sin2

= 2coscos − 2sin2

A + B + C = 180

 = 90 – 

sin = sin (90 – )

sin = sin 90 cos - cos 90 sin

sin = cos

2coscos - 2sinsin

= 2coscos - 2sincos

= 2sincos - 2 sincos

= 2sin[cos - cos]

= 2sin[-2sinsin]

= 2sin[2sinsin]

= 4sinsinsin

 cos *A* + cos *B* + cos *C* – 1 = 4sinsinsin

**Example VI (UNEB 2007)**

Show that 

***Solution***











But 









**Example VII (UNEB Question)**

Show that .

***Solution***





*A + B* = 6*θ* …………………….. (i)

**

*A* – *B* = 6*θ* ……………………… (ii)

Solving Eqn (i) and Eqn (ii) simultaneously;

*A* = 9θ, *B* = 3*𝜃*









**Example VIII (UNEB Question)**

If *A*, *B*, *C* are angles of the triangle, show that

cos 2*A* + cos 2*B* + cos 2*C* = -1 − 4 cos*A* cos *B* cos *C*.

***Solution***

cos 2A + cos 2B + cos 2C

2cos(A + B) cos(A – B) + 2cos2C – 1

= -1 + 2cos(A + B) cos (A – B) + 2cos2C

A + B + C = 180

A + B = (180 – C)

cos(A + B) = cos(180 – C)

cos (A + B) = cos 180 cos C + sin 180 sin C

= -cos C

* -1 + 2cos(A + B) cos (A – B) + 2cos2A

= -1 – 2cos C cos(A – B) + 2cos2C

= -1 – 2cos C[cos(A – B) − cos C]

= -1 – 2cos C[cos(A – B) – cos C]

cos *C* = -cos(*A* + *B*)

= -1 – 2cos C[cos(A – B) + cos (A + B)

= -1 – 4cos A cos B cos C.

cos 2*A* + cos 2*B* + cos 2*C* = -1 − 4 cos*A* cos *B* cos *C*.

**Example IX (UNEB Question)**

Use the factor formula to show that 

***Solution***



= 



= tan(A + B)



**UNEB 2008**

**(i)** Prove that 

**(ii)** Deduce that where A, B and C are ***solution***

**(i)** 



**(ii)** A + B + C = 1800

A + B =180 – C







**Example X (UNEB Question)**

Solve sin*x* – sin 4*x* = sin2*x* – sin3*x* for 

***Solution***

sin *x* – sin 4*x* = sin 2*x* – sin 3*x*

sin 3*x* + sin *x* = sin 4*x* + sin 2*x*









Taking cos = 0





Taking sin 2*x* – sin 3*x* = 0

sin 3x – sin 2*x* = 0





Either 







**Or** Sin



*x* = 0,  are the solutions to the equation

**Relationship between sides of a triangle**

In a triangle ABC with angles A, B and C, we denote the side opposite these angles by their corresponding small letters a, b, and c respectively as shown in the figure below.

*c*

*a*

*b*

*A*

*A*

*B*

*B*

*C*

*C*

**The sine rule**

Let O be the centre of the circle circumscribing the triangle ABC with radius, *R*.

*a*

*c*

*b*

*D*

*C*

*B*

*A*

2*R*

*O*

*A*

Figure I

*A*

*c*

*b*

*a*

*D*

*B*

*A*

*C*

2*R*

*O*

*C*

Figure II

*C*

*b*

*a*

*c*

*D*

*A*

*C*

*B*

2*R*

*O*

*B*

Figure III

*B*

From figure I, BCD = 90°

Since this angle is subtended by the diameter,

 from figure I.

 …………………. (i)

From figure II;

sin C = 

 ……………………… (ii)

From figure III;



2*R* =  …………………….. (iii)

Equating equations (i), (ii), and (iii)



This is the sine rule

**The Cosine rule**

Consider a triangle ABC. Assume angle A is acute.

*h*

*a*

*b – x*

*C*

*B*

*A*

*x*

*c*

*D*

*A*

Considering the right-angled triangle BDA,

*x*2 + *h*2 = *c*2 …………………..…. (i)

from the right-angled triangle BCD,

*a*2 = (*b* – *x*)2 + *h*2

*a*2 = *b*2 – 2*bx* + *x*2 + *h*2 ………….. (ii)

From Eqn (i);

*h*2 = *c*2 – *x*2 …………………..…. (iii)

Substituting Eqn (iii) in Eqn (ii)

*a*2 = *b*2 – 2*bx* + *x*2 + *c*2 – *x*2

*a*2 = *b*2 – 2*bx* + *c*2 ……………….. (iv)

From triangle ABD;



*x* = *c* cos *A* …………..………….. (v)

Substituting Eqn (v) into (iv)

*a*2 = *b*2 – 2*bc* cos *A* + *c*2

*a*2 = *b*2 + *c*2 – 2*bc* cos *A*

**Application of cosine and sine rules**

**Example I**

Prove that in a triangle ABC, 

***Solution***

From the sine rule; 

*a* = 2*R* sin *A*, *b* = 2*R* sin *B* and *c* = 2*R* sin *C*







*A + B + C =* 180

*C* = 180 – (*A* + *B*)

sin *C*  = sin(180 – (*A* + *B*))

sin *C* = sin 180 cos (*A*+*B*) – cos 180 sin (*A*+*B*)

= sin (*A* + *B*)



= 



= 

**Example II**

Prove that in any triangle ABC, = tan*B* cot *C*

***Solution***

From the cosine rule;

*a*2 = *b*2 + *c*2 – 2*bc* cos*A* ……………… (i)

*b*2 = *a*2 + *c*2 – 2*ac* cos *B* ……………… (ii)

*c*2 = *a*2 + *b*2 – 2*ab* cos *C* ……………… (iii)

From Eqn (i);

2*ac* cos *B* = *a*2 + *c*2 – *b*2

From Eqn (iii);

*2ab* cos *C* = *a*2 + *b*2 – *c*2



But from the sine rule;



*a* = 2*R* sin*A*, *b* = 2*R* sin *B*, and *c* = 2*R* sin *C*



= tan B × cot C

= tan*B* cot *C*

**Example III**

Prove that in any triangle ABC, 

***Solution***



From the sine rule; 

*a* = 2*R* sin A, *b* = 2Rsin *B*, *c* = 2*R* sin *C*

 = 

= 

But *A* + *B* + *C* = 180

*A* = 180 – (*B* + *C*)



cos = cos(90 – )

= cos 90 cos + sin 90 sin

= sin

sin = sin(90 – )

= sin 90 cos − cos 90 sin

= cos 





**Example IV**

Prove that in any triangle *ABC*, 

***Solution***

From the sine rule,  = 2*R*

*a* = 2*R* sin*A*, *b* = 2*R* sin *B*, and *c* = 2*R* sin *C*



= 

= 

= 

= 

= 

= 

= 

From triangle ABC;

*A* + *B* + *C* = 180

*A* = 180 – (*B* + *C*)

sin *A* = sin(180 – (*B* + *C*))

= sin 180 cos B + C – cos 180 sin(B + C)

= sin (B + C)





**Area of a triangle**

Let D denote the area of a triangle ABC, then

*D* = 



*D* = *bc* sin  cos

*S* = 

Where *S* is the semi perimeter.

From the cosine rule, *a*2 = *b*2 + *c*2 – 2*bc* cos *A*

cos *A* = 

cos *A* = 1 – 2 sin2

sin2 = 

sin2 = 

sin2 = 

sin2 = 



*a + b + c =* 2*s*

*a + b* – *c* = *a* + *b* + *c* – 2*c*

*=* 2 *s* – 2*c*

*=* 2(*s* – *c*)

*a + c – b* = *a* + *b* + *c* – 2*b*

= 2 *s* – 2b

= 2(*s* – *b*)





From the cosine rule, *a*2 = *b*2 + *c*2 – 2*bc* cos *A*

cos *A* = 

cos *A* = 2 cos2 − 1

cos2 = (1 + cos *A*)



*a* + *b* + *c* = 2*s*

*b + c – a = a* + *b* + *c* – 2*a*

*=* 2*s –* 2*a*

*=* 2(*s* – *a*)



From the area of a triangle *D*;

*D* = *bc* sincos



The area of a triangle is 

This is called the Heron formula named after the Greek Mathematician Heron

**Differentiation and integration of trigonometric functions**

|  |  |  |
| --- | --- | --- |
| **Function** | **Differentiate** | **Integrate** |
| Sin *x* | Cos *x* | −cos*x* |
| cos *x* | -sinx | sin *x* |
| Sin ax | a cos ax | cos *ax* |
| cos *ax* | -*a* sin *ax* | sin *ax* |
| sin 3*x* | 3 cos 3*x* | cos 3*x* |
| cos 3*x* | −3 sin 3*x* | sin 3*x* |

### Differentiation of trigonometric functions

**Example I**

Differentiate the following

1. sin 6*x*
2. *−*3 cos 5*x*
3. −4 sin*x*
4. sin *x*2
5. 2sin(*x +* 1)

***Solutions***

**(a)**  *y* = sin 6*x*



**(b)** *−*3 cos 5*x*

*y = −*3 cos 5*x*



**(c)** −4 sin*x*

*y* = −4 sin*x*



**(d)**  sin *x*2

*y* = sin *x*2



**(e)** 2sin(*x +* 1)

*y* = 2sin(*x +* 1)



**Example II**

Differentiate the following

1. sin2*x*
2. 4cos2*x*
3. cos3*x*
4. 2sin3*x*
5. 3 sin42*x*
6. 

***Solutions***

**(a)** sin2*x*

*y* = sin2*x*



**(b)** 4cos2*x*

*y* = 4cos2*x*



**(c)** cos3*x*

*y* = cos3*x*



**(d)**  2sin3*x*

*y* = 2sin3*x*





**(e)** 3 sin42*x*

*y* = 3 sin42*x*



**(f) **



**Example II**

Differentiate the following

1. *x* cos *x*
2. ***x*** sin 2*x*
3. *x*2sin *x*
4. ****
5. ****
6. ****

***Solutions***

(a) *y* = *x* cos *x*

From *y = uv*;

****

** = *x*(-**sin*x*) + cos *x*

 = -*x* sin *x* + cos *x*

**(b) *x*** sin 2*x*

*y* = *x*sin 2*x*

 = *x* .2cos 2*x* + sin 2*x*.1

 = 2*x* cos 2*x* + sin 2*x*

(c) *x*2sin *x*

*y* = *x*2sin *x*



*x*2cos *x* + (sin *x*) 2*x*

*x*2 cos *x* + 2*x* sin *x*

**(d) **

*y* = 

*y* = 





**(e) **

***y* = **

From *y* = ;





**

**(f) **

*y* = 

From *y* = ;





**Derivatives of tan *x*, cot *x*, sec *x*, and cosec *x***

**(i)** (tan *x*) = sec2*x*

(sec *x*) = sec *x* tan *x*

(cot *x*) = -cosec2 *x*)

(cosec *x*) = -cosec *x* cot *x*

**Proofs**

(tan *x*) = 

= 

= 

= 

(tan *x*) = sec2*x*

(cot *x*) = 

= 

= 

= 

= -cosec2 *x*

 = -cosec2 *x*

**(iii) (sec x)**

= 





= 

= sec *x* tan *x*

**(sec *x*) = sec *x* tan *x*

**



= -cot *x* cosec *x*

**Example I**

Differentiate the following

1. tan 2*x*
2. cot 3*x*
3. 2cosec*x*
4. –tan (2*x* + 1)
5. sec(3*x* – 2)
6. tan

***Solution***

**(a)** tan 2*x*

*y =* tan 2*x*

2sec22*x*

**(b)** cot 3*x*

*y* = cot 3*x*

3(-cosec23*x*)

= -3cosec23*x*

**(c)** 2cosec*x*

*y* = 2cosec*x*





**(d)** –tan (2*x* + 1)

***y* =** –tan (2*x* + 1)

= -2sec2(2*x* + 1)

**(e)** sec(3*x* – 2)

*y* = sec(3*x* – 2)

**

= sec(3*x* – 2) tan(3*x* – 2)

**(f)** tan

*y* = tan



**Example II**

Differentiate the following:

1. *x* tan *x*
2. sec *x* tan *x*
3. *x*2cot *x*
4. 3*x* cosec *x*
5. cosec *x* cot *x*
6. 

***Solutions***

(a) *x* tan *x*

*y* = *x* tan *x*

 = *x* sec2*x* + (tan *x*).1

 = *x* sec2*x* + tan *x*

**(b)** sec *x* tan *x*

*y* = sec *x* tan *x*

 = sec *x* sec2*x* + tan *x*(sec *x* tan *x*)

 = sec3*x* + tan2xsec *x*.

**(d)** 3*x* cosec *x*

*y =* 3*x* cosec *x*

 = 3*x*(-cosec *x* cot *x*) + cosec3

 = -3*x* cosec *x* cot *x* + 3cosec *x*

(e) cosec *x* cot *x*

*y* = cosec *x* cot *x*

 = cosec *x*-cosec2*x* + (cot *x*)(-cot *x* cosec *x*)

= cosec3*x* – cot2*x*cosec *x*

**Example III**

Differentiate the following

1. tan2*x*
2. sec2*x*
3. 3cosec2*x*
4. –tan22*x*
5. cot23*x*
6. 
7. -2cosec4 *x*

***Solution***

**(a)** tan2*x*

*y* = tan2*x*

 = 2tan *x*(sec2*x*)

 = 2sec2*x* tan *x*

**(b)** sec2*x*

*y* = sec2*x*

 = 2sec *x*(sec *x* tan *x*)

 = 2sec2*x* tan *x*

**(c)** 3cosec2*x*

*y* = 3cosec2*x*

 = 3 × 2cosec *x*(-cosec *x* cot *x*)

 = -6cosec2*x*cot *x*

**(d)** -tan22*x*

*y* = -tan22*x*

 = -2(tan 2*x*)(2sec22*x*)

= -4sec22*x* tan 2*x*

**(e)** cot23*x*

*y* = cot23*x*

 =  × 2 cot 3*x*(-3cosec23*x*)

= -3cosec23*x*cot 3*x*

**(f) **



**(g)** -2cosec4*x*

*y* = -2cosec4*x*

 = -8cosec3*x*(-cosec *x* cot *x*)

 = 8cosec4 *x* cot *x*

### Integration of Trigonometric functions

Integration is the process of obtaining a function from its derivative

|  |
| --- |
| **Note:** sin (*ax*) + *c*  cos(*ax*) + *c* |

**Example I**

Integrate the following

1. cos 3*x*
2. sin 3*x*
3. cos(3*x* – 1)
4. sin(2*x* + 1)
5. 6 cos 4*x*

***Solution***

**(a)** cos 3*x*

*y* = cos 3*x*



= 



**(b) **sin 3*x* + *c*

= cos 3*x* + *c*

**(c) **sin(3*x* – 1) + *c*

**(d) ***dx* = cos(2*x* + 1) + *c*

**(e) ***dx* = *dx*

**=  +** *c*

**= **sin 4*x* + *c*

**Example**

Integrate the following

1. sec22*x*
2. 3sec *x* tan *x*
3. –cosec2*x*
4. cosec 3*x* cot 3*x*
5. 2sec2*x* tan *x*
6. 
7. 
8. 

***Solution***

|  |
| --- |
| Note: (tan *x*) = sec2*x*  = tan *x + c*  = sec *x* tan *x*  = sec *x* + *c*  = -cosec2*x*  = -(cot *x*) + c  = -cosec *x* cot *x*  = -cosec *x + c* |

**(a) **

Let *u* = 2*x*

*du* = 2*dx*

*dx* = 

**

**(b) **



= 3 sec *x* + *c*

**(c) **

Let *u* = 



*dx* = 2 *du*

**

**(d) **

Let *u* = 3*x*

*du* = 3 *dx*

**

**** = 

= 

= 

= (-cosec *u*) + *c*

= cosec 3*x* + *c*

**(e) **

Consider (sec2*x*) = 2sec *x*(sec *x* tan *x*)

= 2 sec2*x* tan *x*

 = sec2*x* + *c*

**(f) **



= sec *x + c*

**(g)**  = 

Let *u* = 2*x*

*du* = 2 *dx*

**



**(h)** 



Let *u* = 2*x*

*du* = 2 *dx*

**





**Example III**

Evaluate the following

1. 
2. 
3. 

***Solution***

**(a)**  



**(b)** 



**(c)** 

From cos 2*x* = 1 – 2sin2 *x*

sin2*x* = (1 – cos 2*x*)



**Example**

A particle moves in a straight line such that its velocity in m/s after passing through a fixed point O is 3cos *t* – 2sin*t*. Find:

1. Its distance from O after s
2. Its acceleration after *π* s
3. The time when its velocity is first zero.

***Solution***

*V* = 3cos *t* – 2 sin *t*

**= 3cos *t* – 2 sin *t*

*dS* = (3cos *t* – 2sin *t*) *dt*

*S =* 3 sin *t +* 2 cos *t + c*

When *t =* 0, *S* = 0

0 = 3 sin(0) + 2 cos (0) + *c*

-2 = *c*

*S* = 3 sin *t +* 2 cos *t* – 2.

When *t* = ,

*S* = 3sin + 2 cos - 2

*S* = 3 – 2

*S* = 1 m

*V* = 3 cos*t* – 2 sin *t*

*a =*  = -3 sin *t* – 2cos *t*

*a =*  = -3 sin π – 2 cos π

= 2 m/s2

*a* = 2 m/s2

*V =* 3cos *t* – 2 sin *t*

3 cos *t* – 2 sin *t* = 0.

*R* = cos(*t* + α) = 0

*R* = 

**cos(*t* + α) = 0

cos(*t* + 33.7) = 0

cos(*t* + 33.7) = 0

*t* + 33.7 = cos-10

*t* + 33.7 = 90

t = 56.3

*t* = 

*t* = 0.983 s

## Revision Exercise

1. Solve the following for all values of *x* from 0° to 360°.

|  |  |
| --- | --- |
| 3. tan *x* = 1 | 1. tan *x* = –1 2. sin *x* = 3. cos *x* = |

1. Solve the following equations for values of *x* from -180° to 180°
2. 
3. 
4. 
5. 
6. 
7. 
8. Solve the following equations for all values of *x* from 0° to 360°
9. 
10. cos *x* = -0.7
11. tan *x* -0.75
12. cos2*x* = 
13. sin *x* = 2cos *x*
14. 2sin *x* – 3cos *x* = 0
15. sin 2*x* = 
16. cos 2*x* = 
17. sin(*x* + 20) = 
18. tan(*x* – 30) = 1
19. 3(cos *x* – 1) = -1
20. sin *x* (1 – 2cos *x*) = 0
21. cos *x*(2sin *x* + cos *x*) = 0
22. 2sin *x* cos *x* + sin *x* = 0
23. 4sin *x* cos *x* = 3cos *x*
24. 4cos2*x* + cos *x* = 0
25. tan *x* = 4 sin *x*
26. (2sin *x* – 1)(sin *x* + 1) = 0
27. 2sin2*x* – sin *x* – 1 = 0
28. 2tan2*x* – tan*x* – 6 = 0
29. 2tan *x* –  = 1

Solve the following equations for all values of *x* from

-180° to 180°

1. cos2*x* = 
2. sin 2*x* = 2cos 2*x*
3. cos(*x* – 20) = 
4. cos *x*(sin *x* – 1) = 0
5. 3sin2*x* = 2sin *x* cos *x*
6. 2cos2*x* – 5cos *x* + 2 = 0
7. Factorise the expression 6sin*θ* cos*θ* + 3cos*θ* + 4sin*θ* + 2. Hence solve 6sin*θ* cos*θ* + 3cos*θ* + 4sin*θ* + 2 = 0 for -180° ≤ 180°
8. Factorise the equation 3sin*θ* cos*θ* – 3sin*θ* + 2cos*θ* – 2. Hence solve 3sin*θ* cos*θ* – 3sin*θ* + 2cos*θ* = 2.
9. Without using tables or calculator, find the values of:

|  |  |
| --- | --- |
| 1. sec 45° 2. cot 45° 3. cosec 30 4. sec 60° 5. cosec 135° 6. sec 120° | 1. cosec 330° 2. sec 240° 3. cot -135° 4. sec -60° 5. sec(-120°) 6. cosec 315° |

1. Simplify the following expression:
2. 
3. cosec*θ* tan*θ*
4. 
5. cot
6. Prove the following identities
7. sin*θ* tan*θ* + cos*θ* = sec*θ*
8. cosec*θ* – sin*θ* = cot*θ* cos*θ*
9. (sin*θ* + cos*θ*)2 + (sin*θ* – cos*θ*)2*θ* = 2
10. (sin*θ* + cosec*θ*)2 = sin2*θ* + cot2*θ* + 3*θ*
11. cot4*θ* + cot2*θ* = cosec4*θ* – cosec2*θ*
12.  = cosec*θ* – cot*θ*
13.  **=** 2cosec*θ*
14.  = cot*θ*
15.  = tan*θ*
16.  = 2cosec*θ*
17. cos4*x* – sin4*x* = cos2*x*
18. cos *A* + cos(*B + C*) = 0
19. = 2cos*θ*
20. Prove the following identities:
21. 2cosec 2*θ* = cosec*θ* sec*θ*
22. tan *A* + cot *A* = 2coesec 2*A*
23.  = sec 2*A*
24. cot 2*A* = cosec 2*A* – tan *A*
25.  = cot*θ*
26. tan*θ* – cot*θ* = -2cot 2*θ*
27. Prove the following identities:
28.  = tan2*θ*
29. tan 3*θ* = 
30.  = tan *θ*
31. Eliminate *θ* from each of the following pairs of relationships
32. *x* = sin *θ* , *y* = cos *θ*
33. *x* = 3sin *θ* , *y* = cosec *θ*
34. 5*x* = sin *θ*, *y* = 2cos *θ*
35. *x* = 3 + sin *θ*, *y* = cos *θ*
36. *x* = 2 + sin, cos *θ* = 1 + *y*.
37. Solve the following equations for all values of *θ* from -180° to 180°.
38. 4 – sin *θ* = 4cos2 *θ*
39. sin2*θ* + cos *θ* + 1 = 0
40. 5 – 5cos *θ* = 3sin2*θ*
41. 8tan *θ* = 3cos *θ*
42. sin2*θ* + 5cos2*θ* = 0
43. 1 – cos2*θ* = -2sin *θ* cos *θ*
44. Solve the following equations from 0° to 360°
45. sec *θ* = 2
46. cot 2*θ* = 
47. 3cot *θ* = tan *θ*
48. 2sin *θ* = -3cot *θ*
49. 2sec2*θ* – 3 + tan *θ* = 0
50. If *A* + *B* + *C* = 180°, prove that

cos2*A* + cos2*B* + cos2*C* = 1 – cos*A*cos*B*cos*C*

1. Prove that sin 3A = 4sinAsin(60 + A)sin(60 – A)
2. Show that in a triangle *ABC*, if 2*S* = *a* + *b* + *c*, then

1 – tan + tan = 

1. Prove that in any triangle *ABC*,

(*a* + *b* + *c*)(tan + tan) = 2c cot.

1. Prove that in any triangle

*ABC*,  = tantan

1. From a point A, a light wind due to north of A has an elevation α from a point B, due west of A. The angle of elevation is β. Prove that the angle of elevation from the midpoint of *AB*. is 
2. Solve: 4cos α – 3sin α = 2
3. Solve the equation 15cos2*θ* + 20sin2*θ* + 7 = 0
4. Find all the possible values of *x* that satisfy 
5. Prove that 
6. Solve the equation  = 0 for 0 ≤ *x* ≤ 2π.
7. Solve cos4*x* + sin4*x* =  for 0 ≤ *x* ≤ .
8. Find the value of *x* for 3cos2*x* – 8cos*x* + 4 = 0
9. Show that 
10. Prove that 
11. Solve the equation cos *x* – cos 4*x* = cos 2*x* – cos 3*x* for –*π* ≤ *x* ≤ *π*.
12. Given that *y* = 4cos *x* – 6sin *x*. Express *y* in the form *R*cos(*x* + *α*), where *R* is a constant. Find the maximum and minimum value of *y*.
13. Express (45° + *x*) in terms of tan *x.* Hence or otherwise express tam 75° in the form *a* + *b*.
14. Given sin *x* = , where 180° ≤ *x* ≤ 270, find without using tables or calculator the value of tan 3*x*.
15. Show that:
16. 
17. 
18. cos-1*x* + sin-1*x* = 
19. Solve the equation
20. tan-1(2*x* + 1) + tan-1(2*x* – 1) = tan-12
21. tan-1(1 + *x*) + tan-1(1 – *x*) = 32
22. cos-1*x* + cos-1*x* = 
23. 
24. Without using tables or calculator, evaluate 

6. ±1, ±*i*, (1 – *i*), (1 + *i*)

7. (i) -1 + *i*, 2, 2*π*/3;  + *i*, 1, π/6

(ii) 1 ±2*i*; (a) *p*2 = *q* – 4, (b) 2*p* = *q* + 5

8. (a) .

9. (i) -1 – *i*, 3π/4, (ii) 2 – *i*, 2; -10.

10.(b) -1, , (c) -1 – 3*i*, 

11. , ; 28

12. (ii) *x*3 – 3*abx* + *a*3 + *b*3; , , 

13. (i) ±(3 – 2*i*), (ii) 3 ±.

14. (i) (a) 7/5, -4/5; (b) 2 ± *i*

15. (i) 1 – *i*, -1 ± 2*i*, (ii) ; π/6, 2π/3

16. (i) 2, 32, π/3, -π/3; , (ii) 2 + 3*i*

17. (i) -1, (ii) 1 + 2*i*, 

18. (i) , -π/4; , 8*i*.

20. (i) 2 – *i*, 3 – 4*i*; (2, -1)

21. (ii) 3*x*2 + 3*y*2 + 10*x* + 3 = 0.

22. (i) *x* = 1, *y* = 2 or *x* = -1, *y* = -2

23. (i) a) 13, -23°, (b) 1, 90°; (ii) 12;

(iii) |*z* + 1 + *i*| ≥ 4

24. ; . 25. .

# VECTORS

***Straight line in space***

A straight line is uniquely determined in space if either; we know one point on the straight line and its direction or two points on the straight line.

***Vector equation of a line***

The vector equation of a line is given by

*r* = ***a*** + *λAB*

B

*r* = *a* + λ*AB*

*A*

*r* = ***a*** + *λAB*

*r = a +* λ*d*

Where; *a* = any point on the line

*d* = directional vector of the line.

|  |
| --- |
| The Cartesian equation is given by;  Where a, b and c are direction vectors |

**Example 1**

Find the vector and Cartesian equation of a line passing through and is parallel to

Solution;

Cartesian equation

**Example II**

Find the vector and the Cartesian equation of a line passing through A(3, 4, -7) and B(1, -1, 6)

***Solution***

(vector equation of line)

*x* = 3 – 2λ

*y =* 4 - 5λ

*z* = -7+13λ

**

Cartesian equation

**Example III**

Find the vector and Cartesian equation of a line passing through (2, -1, 1) and is parallel to the line whose equation

***Solution***

*Since the lines are parallel, it implies that they have the same parallel vectors*.

Cartesian equation:

*x* – 2 = 2λ 

*y* + 1 = 7λ 

*z* – 1 = -3λ 

**

**Example III**

Find the vector and Cartesian equations of the a line passing through the following points

1. 5, -4, 6) and (3, 7, 2)
2. (3, 4, -7) and (5, 1, 6)

***Solution***

*r* = *a* + λ*AB*

*r* = *a* + λ*d*



1. A(3, 4, -7) and B(5, 1, 6)

*r* = *a* + *μd*



**Example IV**

Find the coordinates of the point where the line joining the points (2, 3, 1) and (3, -4 -5) meets the *x-y* plane

For the line to meet the *x-y* plane, *z* = 0

The coordinates are

**Example V**

Show that 4**i** – **j** – 12**k** lies on the line

***Solution***

and

∴ The point lies on the line since the values of *μ* are the same.

**Example V**

The points A, B, C have position vectors Find which of the three points lie in the line

***Solution***

For *A*,

4 = -1 + 3λ  λ = -1

5 = 4 – λ  λ = -1

-1 = 1 + 2λ  λ = -1

lies on the line.

For B,

5 = -1 + 3λ  λ = 2

2 = 4 – λ  λ = 2

3 = 1 + 2λ  λ = 1

Since the values of λ are not the same, point B does not lie on the line.

For C,

8 = -1 + 3λ  λ = 3

1 = 4 – λ  λ = 3

7 = 1 + 2λ  λ = 3

Since the vales of λ are the same, point C lies on the line.

**Angle between two lines**

The angle between two lines is the angle between their directional vectors

|  |
| --- |
| **Consider two lines *L*1 and *L*2 with vector equations**  ***r* = *a* + λ*d*1 and *r* = *b* + μ*d*2 respectively**  **The angle between the two lines is given by the formula** |

**Examples**

1. Find the angle between the lines;

***)***

***)***

**Example II**

Find the angles between the lines

***Solution***

,

**Example III**

Find the acute angle between the lines:

 and 

***Solution***

 and 

and



The acute angle between the two lines is 47.1°

**Example IV**

Find the angle between the lines:

 and 

***Solution***

and



θ = 40.2°

The acute angle between the two lines is 40.2°

|  |
| --- |
| **Note:** **If two lines are perpendicular, then (** |

**Point of Intersection of two Lines**

**Example**

Find the point of intersection of the lines

***Solution***

………………. (i)

……………. (ii)

From equation (i)

*x* = λ ………………………………. (iii)

………………………. (iv)

………………………. (v)

From equation (ii)

………………………. (vi)

*z* = μ + 4 …………………………... (viii)

Substituting Eqn (\*) in Eqn (\*\*)

Equating Eqn (v) and Eqn (viii)

The two lines intersect

=1

The point of intersection of the lines is (4, 6, 1)

**Example II**

Find the point of intersection of the line

*)*

*)*

***Solution***

From )

…………………… (1)

*)*

Equating the corresponding *x* components:

Equating the corresponding *y* components:

Equating the corresponding *z* component;

……………………….. (5)

Eqn (3) –eqn (4)

From Eqn (4)

Substituting and in Eqn (5);

The two lines intersect at (5, 0, 1)

**Example III**

Find the point of intersection of the lines

***Solution***

From equation (\*)

From equation (\*\*)

Equating the corresponding components

Eqn(8) - (7)

-5

Substitute λ = 2 in Eqn (8)

The point of intersection is (4, 5, 9)

## PLANES

A plane is a surface which contains at least three non-collinear points. If two points are taken then the lines joining the two lines lies completely on the surface of the plane.

A plane is completely known if we know one point that lie on the plane and then the normal to the plane.

**Equation of a Plane**

Suppose a plane P passes through a point A with a position vector ***a*** and is perpendicular to vector **n**. Let **r** be any point (*x*, *y*, *z*) in the plane.

If two lines are perpendicular, dot product of their direction vector = 0

**n**

R

A

***a***

r

O

|  |
| --- |
| Equation of a plane is given by |

Where **n** = normal and **a** = the point that lies on the plane.

**Example** **I**

Find the equation of a plane passing through (1, 2, 3), and is perpendicular to vector

***Solution***

**Example II**

Find the equation of a plane which contains A with position vectorand is perpendicular to **.**

***Solution***

**Example III**

Find the equation of a plane passing through a point A with a position vector and is perpendicular to the vector**.**

***Solution***

**Angle between two planes**

The angle between two planes is the angle between their normals

|  |
| --- |
|  |

**Example I**

Find the angle between the planes

***Solution***

,

**Example II**

Find the angle between the planes 3*x* – 3*y* – *z* = 0 and

***Solution***

,

**Angle between a line and a plane**

α

**r**=*a*+λ**d**

*n*

θ

Line

|  |
| --- |
|  |

***Example***

Find the angle between the lines

*)* and the plane

***Solution***

Find the acute angle between the line  
 and

***Solution***

***Solution***

Find the angle between the line and

***Solution***

**Point of intersection of a line and a plane**

**Example** I

Find the point of intersection of the line and

***Solution***

 ………………. (\*)

From (\*)

……………….. (1)

……………….…. (2)

………………..... (3)

From equation (1)

From equation (2)

From equation (3)

The point of intersection

**Example II**

Find the point of intersection of the line and the plane

***Solution***

 ……………….. (\*)

………………………….. (1)

……………………... (2)

…………………….. (3)

The point of intersection = (5, 0, 5)

***Example***

Find the point of intersection of the line; and the plane

***Solution***

…………….. (1)

……………... (2)

………………. (3)

The point of intersection (-6, -6, 0)

**Perpendicular distance of a point from a plane**

The perpendicular distance of a point from the plane is given by the formula;

***Example***

Find the distance of a point from the plane

***Solution***

Comparing with

;

Units

**Line of intersection of two planes**

Two planes intersect in a line

**Examples I**

Find the line of intersection of the planes 2*x* + 3*y* + 4*z* = 1 and *x* + *y* + 3*z*=0

***Solution***

Let

…………………….. (1)

Eqn (2) ×2

………………………. (3)

Eqn (1) – Eqn (3);

From Eqn (2);

But *y* = 1 + 2

*x* + *y* = -3λ

*x* + 1 + 2λ = -3λ

**Example II**

Find the line of intersection of planes 2*x* + 3*y* – *z* = 4 and *x* – *y* + 2*z* = 5.

***Solution***

Let

……………… (i)

……………….. (ii)

Multiply Eqn (ii) by 3;

…………… (iii)

Eqn (iii) + Eqn (i);

Multiply Eqn (ii) by 2;

…………… (iv)

Eqn (iv) – Eqn (i);

**Example**

Find the Cartesian equation of a line of intersection of the lines.

Let

…………… (i)

…………… (ii)

Eqn (i) × 2

………….. (iii)

Eqn (iii) + Eqn (ii)

Eqn (i) × 4

……….. (iv)

Eqn (ii) × 3

………………. (v)

Eqn (iv) + Eqn (v)

**Equation of a Plane**

Given three points on the plane, we can find the equation of a plane;

**Example I**

Find the Cartesian equation of a plane passing through A (0, 3, -4) B (2, -1, 2) and C (7, 4, -1)

***Solution***

Let the normal

…………………… (i)

…………………… (ii)

From (i)

………………………… (iii)

……………………………… (iv)

**Example II**

Find the equation of a plane passing through points P(4, 2, 3), Q(5, 1, 4) and R(-2, 1, 1).

***Solution***

Let the normal to the plane be

…………………… (i)

………………… (ii)

From Eqn (i);

**Example III**

Find the equation of the planes passing through the following points:

**(i) A (0, 2, -4) B (2, 0, 2) C (-8, 4, 0)**

***Solution***

Let the normal

*p* – *q* + 3*r* = 0 …………………… (i)

-4*p* + *q* + 2*r* = 0 …………………… (ii)

**(ii) A (-1, 0, 1), B(3, 3, -2), C(-1, 1, 1)**

Let the normal

………………… (i)

Substitute *q* = 0 in Eqn (i);



**Example IV**

Find the Cartesian equation of a plane containing the point (1, 3, 1) and it’s parallel to vectors (1, -1, -3) and (2, 1, -3)

***Solution***

and

Let the normal

……………….. (i)

……………… (ii)

*q* = 3*r*

*p* = 3*r* – 3*r*

*p =* 0



**Example V**

Find the Cartesian equation of the plane passing through the points A(1, 0, -2), B (3, -1, 1) parallel to the line

**Solution:**

2*p* – *q* + 3*r* = 0 …………………….. (i)

………………………… (ii)

From Eqn (ii);

**Example VI**

Find the equation of the plane containing line

and is parallel to the line

−2*p* + *q* – *r* = 0

⇒2*p* – q + *r* = 0 ……………….. (i)

-*p* + *q* + 2*r* = 0……………….. (ii)

From Eqn (i);

*r* = –2*p* + *q*

⇒ *p* – *q* – 2(*q* – 2*p*) = 0

*p* – *q* – 2*q* + 4*p* = 0

5*p* – 3*q* = 0

*p* = 



3*x* + 5*y* – *z* = 3 – 5 + 0

3*x* + 5*y* – *z* = -2

**Example VII**

Find the Cartesian equation of the plane formed by the lines **r** = -2**i** + 5**j** – 11**k** + λ(3**i** + **j** + 3**k**) and

**r** = 8**i** + 9**j** + λ(4**i** + 2**j** + 5**k**)

***Solution***

3*p + q +* 3*r* = 0 …………….. ….(i)

4*p* + 2*q* + 5*r* = 0 ……………… (ii)

From Eqn (i);

*q* = -3*p* – 3*r*

4*p* + 2(-3*p* – 3*r*) + 5*r* = 0

4*p* – 6*p* – 6*r* + 5*r* = 0

-2*p* – *r* = 0

*r* = -2*p*

*q* = -3*p* – 3(-2*p*)

*q* = 3*p*

*x* + 3*y* – 2*z* = -2 + 15 + 22

*x* + 3*y* – 2*z* = 35

***INTERNAL AND EXTERNAL DIVISIONS***

Let A and B be points in space with position vectors A and B.

0

B

A

R

***b***

***a***

*μ*

𝜆

Let R be a point on a line segment AB dividing AB internally in the ratio of

**OR = OA + AR**

**Example I**

Given that; . Find the coordinates of *R* such that *PR* :*RQ* = 1:2

**Example II**

The points A and form a line segment which is divided externally in the ratio of 4:-1. Find the coordinates of T

**Example III**

Find the position vectors and , Find the position vectors of C which divides AB externally in the ratio of 5:-3

***Solution***:

**Example IV**

Given that A(0, 5, -3), B(2, 3, -4) and C(1, -1, 2). Find the coordinates of D if ABCD is a rectangle or parallelogram.

D

C (1, -1,2)

B (2, 3, -4)

A (0, 5, -3)

**Proving that three points are vertices of a triangle**

Give a triangle ABC with vertices B (x2, y2, z2) C (x3, y3, z3)

C

A

B

|  |
| --- |
|  |

**Example**

Show that  **and**

are vertices of a triangle

C(11, 4, 5

A(3, 3,1)

B(8, 7, 4

**Length and the equation of the perpendicular drawn from the point**

**Example I**

Find the equation and length of the perpendicular drawn from a point (2, 3, -4) to the line

***Solution***

A(2, 3, -4)

B(4-2λ), 6λ, (1-3λ)

=0

Equation of the perpendicular



Length of the perpendicular AB

*AB =* 3.1719 units

Find the length and equation of the perpendicular drawn from a point (5, 4, -1) to the line;

***Solution***

A(5, 4, -1)

B(1+2λ, 9λ, 5λ)

units

Equation of the perpendicular bisector is

**Shortest Distance between Parallel Planes**

**Example I**

Find the perpendicular distance between two parallel planes;

***Solution***



Plane 1

Plane 2

2 units

1 unit

***O***

**Example II**

Find the perpendicular distance between two parallel planes;



For plane 1



For plane 2

0

units

unit

***O***



**Shortest distance between two parallel lines**

θ

θ

A

B

d

O

d



B

O

A

Distance between a point A and line B



**Example I**

Find the shortest distance between the following pairs of parallel lines

 and







θ

θ

A (2, 1, 3)

B (-1, 3, 1)



d



s

**Example II**

Find the distance between the following pairs of parallel lines

***Solution***

θ

θ

A (2, 0, 3)

B (1, -1, 4)

d



***SKEW LINES***

These are lines which are neither parallel nor perpendicular

Shortest distance between two skew lines

**Example I**

Find the shortest distance between the following skew lines

and

A (-1+λ,2+2λ+3+λ)

B(2μ, -1+μ, 1+3μ)















***Example II***

Find the shortest distance between the following pairs of skew lines



***Solution***

 , 

A (2, -1+λ,2λ)

B(-1+μ, 1-3μ, 1-2μ)

**Vector Geometry**

**Example I**

Triangle OAB has OA=**a,** OB=**b**. C is a point on OA such that OC=**a**. D is a mid point of AB when CD is produced, it meets OB at E such that DE = nCD and BE=**k**b. Express BE, DE in terms of;

a) **n, a** and **b**

b) **k, b** and **a**. Hence find the values of **n** and **k.**



B

E

D

O

C

A







**Example II**

Given that OA is **a** and OB=**b** point R is on OB such that OR:RB=4:1. Point P is on AB such that BP:PA=2:3. When RP and OA are both produced, they meet at Q. Find OR and OP in terms of **a** and **b**

ii) OQ in terms of **a**

***Solution***

***Q***

***O***

***B***

***P***

***R***

***a***

***b***

***A***











OQ = 

**Example III**

O, A and B are non collinear points OA = a, OB = b, C is midpoint of AB, D is a point on OB such that . T is a point of intersection of OC and AD. Find the vector OT in terms of *a* and *b*.

***Solution***

***O***

***B***

***C***

***D***

***a***

1

***A***

3

T

**OT =** λOC

**OC = OB + BC**

**= b + **BA

**= b** + ****(**a – b)**

**OC = **(**a + b)**

****

**OT** = **OA + AT**

**= a + AD**

**AD = AO + OD**

**= a + b**

**OT = a + **(**a + b**)

**OT = a − a + b**

**OT =** (1 **− **)**a + b** ……..….. (ii)

Equating components of vectors **a** and **b** in Eqns (i) and (ii);

****

From Eqn (iv);





****

# Revision Exercise

1. In a triangle *ABC*, the altitudes from B and C meet the opposite sides at E and F respectively. BE and CF intersect at O. Taking O as the origin, use the dot product to prove that *AO* is perpendicular to BC

(b) Find the point of intersection of the line  with the plane

3*x* + 4*y* + 2*z* – 25 = 0

(c) Find the angle between the line  and the plane 4*x* + 3*y* + 1 = 0

1. (a) Show that the equation of the plane through points A with position vector 2i + 2k perpendicular to the vector i + 3j – 2k is *x* + 3*y* – 2*z* + 10 = 0

(b) (i) Show that the vector 2i – 5j + 3.5k is perpendicular to the line **r** = 2i – j + λ(4i + 3j + 2k)

(ii) Calculate the angle between the vector 3i – 2j + k and the line in (b)(i) above.

1. A point P has coordinates (1, -2, 3) and a certain plane has the equation *x* + 2*y* + 2*z* = 8. The line through P parallel to the line  meets the plane at a point Q.
2. (a) The line through A(1, -2, 2) and perpendicular to the plane 4*x* – *y* + 2*z* + 12 = 0 meets the plane in point B. Find the coordinates of B.

(b) Given that the vectors a**i** – 2**j** + **k** and 2a**i** + a**j** – 4**i** are perpendicular, find the values of *a*.

1. Find the equation of the plane through the point (1, 2, 3) and perpendicular to the vector **r** = 4i + 5j + k.
2. (a) The vertices of a triangle are P(2, -1, 5), Q(7, 1, -3) and R(13, -2, 0). Show that = 90°. Find the coordinates of S if PQRS is a rectangle.

(b) Find the equation of the line through A(2, 2, 5) and B(1, 2, 3)

(c) If the line in (b) above meets the line  at P, find the:

(i) coordinates of P,

(ii) angle between the two lines

1. The position vector of points P and Q are 2i – 3j and 3i – 7j + 12k respectively. Determine the length of PQ. PQ meets the plane 4*x* + 5*y* – 2*z* = 5 at point S. Find:
2. the coordinates of *S*,
3. the angle between PQ and the plane.
4. (a) Find the angle between the line **r** = 3k + λ(7i – j + 4k) and the plane **r**(2i – 5j – 2k) = 8

(b) Show that the lines with vector equations

**r**1 = (1 + 4λ)*i* + (1 – λ)j + (2λ)k , and

**r2** = (5 + 3μ)i + (2μ)j + (2 – 5μ)k.

intersect at right angles and give the position vector of the point of intersection.

1. Find the equation of the line with directrix vector **d** which passes through the point with position vector **a** given that

(a) **a** = *i* + 2*j* – *k*, **d** = 3*i* – *k*

(b) **a** = 4*i* – 3*k*, **d** = *i* – 3*j* + 3*k*

1. Find the vector equation of the line which passes through the points with (a) position vectors 3*i* – 3*j* + *k* and -2*j* + *j* + *k*.

(a) position vector *i* + 4*j* and 3*i* – *j* + 2*k*,

(b) coordinates (0, 6, -6) and (5, -7, 2)

(c) coordinates (0, 0, 0) and (5, -2, 3)

1. Write down in parametric form the vector equations of the planes through the given points parallel to the given pairs of vectors.

(a) (1, -2, 0); i + 3j and –j + 2k

(b) the origin; 2i – j and –i + 2j – 7k

(c) (3, 1, -1); j and i + j + k.

1. Find a vector equation for the plane passing through the points with position vectors 2*k*, *i* – 3*j* + *k* and 5*i* + 2*j*.
2. Find the vector equation of the plane through the points A(1, 0, -2) and B(3, -1, 1) which is parallel to the line with vector equation **r** = 3*i* + (2λ – 1)*j* + (5 – λ)*k*. Hence find the coordinates of the point of intersection of the plane and the line **r** = *μi* + (5 – μ)*j* + 2μ – 7)*k*.
3. Find a vector equation for the line joining the points

(a) (2, 6) and (5, 2)

(b) (-1, 2, -3) and (6, 3, 0).

1. (a) Points A and B have coordinates (4, 1) and (2, -5) respectively. Find a vector equation for the line which passes through A and perpendicular to the line AB.

(b) Points P and Q have coordinates (3, 5) and (-3, -7) respectively. Find a vector equation for the line which passes through the point P and which is perpendicular to the line PQ

1. Find a vector equation for the perpendicular bisector of the points:

(a) (6, 3) and (2, -5)

(b) (7, -1) and (3, -3)

1. Points P, Q and R have position vectors 4i – 4j, 2i + 2j, and 8i + 6j respectively.
2. Find a vector equation for the line L1 which is the perpendicular bisector to the points P and Q
3. Find a vector equation for the line L2 which is the perpendicular bisector to the points A and R.
4. Hence find the position vector of the point where L1 and L2 meet.
5. Two lines L1 and L2 have equations  and .
6. Show that L1 and L2 are concurrent (meet at a common point) and find the position vector of their point of intersection.
7. Find the angle between L1 and L2.
8. Points P, Q, and R have coordinates (-1, 1), (4, 6) and (7, 3) respectively.
9. Show that the perpendicular distance from the point R to the point PQ is .
10. Deduce that the area of the triangle PQR is 15 sq.units.
11. Points A, B and C have position vectors –i + 3j + 9k, 5i + 6j – 4k and 4i + 7j + 5k respectively. P is the point on AB such that . Find:

(a) 

(b) 

(c) Find the perpendicular distance from the point C to the line AB.

1. Two lines L1 and L2 have vector equations

**r**1 = (2 – 3λ)i + (1 + λ)j + 4λk

**r**2 = (-1 + 3λ)i + 3j + (4 – λ)k respectively. Find:

**(a)** the position vector of their common point of intersection.

**(b)** the angle between the lines.

1. Find the equation of the plane containing points P(1, 1, 1), Q(1, 2, 0) and (-1, 2, 1).
2. Find the equation of the plane containing point (4, -2, 3) and parallel to the plane 3*x* – 7*z* = 12
3. Show that the point with position vector 7i – 5j – 4k lies in the plane r = 4i + 3j + 2k + λ(i – j – k) + μ(2i + 3j + k). Find the point at which the line *x* = *y* – 1 = 2z intersects the plane 4*x* – *y* + 3*z* = 8.
4. Find the parametric equations for the line through the point (0, 1, 2) that is parallel to the plane *x* + *y* + *z* = 2 and perpendicular to the line *x* = 1 + *t*, *y* = 1 – *t*, *z* = 2*t*.
5. Find the distance between the parallel planes

z = *x* + 2*y* + 1 and 3*x* + 6*y* – 3*z* = 4

1. Two planes are given by the parametric equations

*x* = *r* + 3 and *x* = 1 + *r* + *s*

*y* = 3s and *y* = 2 + *r*

*z* = 2*r* and *z* = -3 + 5

Find the Cartesian equation of the intersection point.

1. The equation of a plane P is given by , where *r* is the position vector of *P*. find the perpendicular distance from the plane to the origin.
2. The line through point P(1, -2, 3) and parallel to the line  meets the plane *x* + 2*y* + 27 8 at Q. find the coordinates of Q.
3. (a) Find the angle between the plane *x* + 4*y* – *z* = 72 and the line **r** = 9i + 6j + 8k.

(b) obtain the equation of the plane that passes through (1, -2, 2) and perpendicular to the line 

(c) Find the parametric equations of the line of intersection of the plane *x* + *y* + *z* = 4 and

*x* – *y* + 2*z* + 2 = 0

1. Find the point of intersection of the three planes 2*x* – *y* + 3*z* = 4, 3*x* – 2*y* + 6*z* = 3 and 7*x* – 4*y* + 5*z* = 11.
2. Find the Cartesian equation of the plane with parametric vector equation 
3. Find the Cartesian equation of the plane containing the point with position vector  and parallel to the vectors  and .
4. Find the Cartesian equation of the plane containing the points with position vectors  and .
5. Find the perpendicular distance from the plane **r**.(2i – 14j + 5k) = 10 to the origin.
6. Find the position vector of the point where the line  meets the plane .
7. Two lines have vector equations  and . Find the position vector of the point of intersection of the two lines and the Cartesian equation of the plane containing the two lines.
8. The position vector of points P and Q are 3i - j + 2k and 2i + 2j + 3k, respectively. Find the acute angle between PQ and the line 1 – *x* = .

(b) Find the point of intersection of the line *x* – 2 = 2*y* + 1 = 3 – *z* and the plane *x* + 2*y* + *z* = 3.

(c) Find the equation of the plane through the origin parallel to the lines **r** = 3i + 3j – k + *s*(i – j – 2k) and **r** = 4i – 5j – 8k + *t*(3i + 7j – 6k)

1. (a) The points A and B have position vectors **a** = 2i – j + 6k and **b** = 7i – 6j + k respectively. Find the coordinates of a point P which divides the vector AB in the ratio:
2. 4:1
3. 1:4
4. (b) Find the Cartesian equation of the plane through the origin parallel to the lines  and 

(c) Find the angle between the line  and the plane

2*x* – 3*y* – 2*z* + 5 = 0.

1. (a) Determine the unit vector perpendicular to the plane containing the points A(0, 2, -4),

B(2, 0, 2) and C(-8, 4, 0).

(b) Find the equation of the plane in (a) above

(c) Show that the point T(5, -4, 3) lies on the plane in (a) above.

(d) Write down the equation in the form **r** = *a* + λb of the perpendicular through the point P(3, 4, 2) to the plane in (a) above.

(e) If the perpendicular meets the plane in (a) above at N, determine vector NP.